1) [12 points] Consider the graph $y = f(x)$ below:

Find [no need to justify]:

(i) $f(2) = 1$

(ii) $\lim_{x \to -2} f(x) = 1$

(iii) $\lim_{x \to 0} f(x) = -\infty$

(iv) $\lim_{x \to 2^-} f(x) = 0$

(v) $\lim_{x \to 2^+} f(x) = 2$

(vi) $\lim_{x \to \infty} f(x) = 1$
2) [28 points] Compute the following limits. [Show work in all!]

(a) \( \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 2x + 1} \)

Solution.

\[
\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 2x + 1} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{(x - 1)^2} = \lim_{x \to 1} \frac{x + 2}{x - 1}
\]

This gives us “3/0”, some some kind of infinite limit. Analyzing the signs we see that \( x + 2 \) is positive on both sides of 1, while \( x - 1 \) is positive on let left and negative on the right of one. This gives,

\[
\lim_{x \to 1^+} \frac{x^2 + x - 2}{x^2 - 2x + 1} = \lim_{x \to 1^+} \frac{x + 2}{x - 1} = +\infty
\]

and

\[
\lim_{x \to 1^-} \frac{x^2 + x - 2}{x^2 - 2x + 1} = \lim_{x \to 1^-} \frac{x + 2}{x - 1} = -\infty.
\]

Thus, \( \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 2x + 1} \) does not exist and is neither +\( \infty \) nor -\( \infty \). [It is a split infinite limit.]

(b) \( \lim_{x \to 4} \frac{x - 4}{\sqrt{x} - \sqrt{8 - x}} \)

Solution.

\[
\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - \sqrt{8 - x}} = \lim_{x \to 4} \frac{x - 4}{\sqrt{x} - \sqrt{8 - x}} \cdot \frac{\sqrt{x} + \sqrt{8 - x}}{\sqrt{x} + \sqrt{8 - x}} = \lim_{x \to 4} \frac{(x - 4)(\sqrt{x} + \sqrt{8 - x})}{x - (8 - x)} = \lim_{x \to 4} \frac{(x - 4)(\sqrt{x} + \sqrt{8 - x})}{2(x - 4)} = \lim_{x \to 4} \frac{\sqrt{x} + \sqrt{8 - x}}{2} = 2
\]
(c) \( \lim_{x \to \infty} \frac{2e^{3x} - 1}{1 - e^x - 3e^{4x}} \)

\[ \lim_{x \to \infty} \frac{2e^{3x} - 1}{1 - e^x - 3e^{4x}} = \lim_{x \to \infty} \frac{e^{3x}(2 - \frac{1}{e^x})}{e^x(e - 3 - \frac{e^x}{e^x})} \]
\[ = \lim_{x \to \infty} \frac{2 - \frac{1}{e^x}}{e^x - 3} \]
\[ = -\frac{2}{3} \]

\[ \square \]

(d) \( \lim_{x \to 1^+} \frac{x + 2}{x^2 - 1} \)

\[ \text{Solution. Trying to evaluate, we get "3/0", so it is [again] some kind of infinite limit. On the left of 1 [i.e., for } x > 1 \text{ but close to 1] we have that both } x + 2 \text{ and } x^2 - 1 \text{ are positive, so} \]
\[ \lim_{x \to 1^+} \frac{x + 2}{x^2 - 1} = +\infty \]

\[ \square \]
3) [15 points] Give the equation of the line tangent to the graph of \( f(x) = \cos(2x) \) at \( x = 0 \). [You cannot use any derivative formula we haven’t seen in class yet!]

Solution. We have

\[
 f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \\
= \lim_{h \to 0} \frac{\cos(2h) - \cos(0)}{h} \\
= \lim_{h \to 0} \frac{\cos(2h) - 1}{h} \\
= \lim_{k \to 0} \frac{\cos(k) - 1}{k/2} \quad \text{[use subst. } k = 2h]\n\]

\[
= \lim_{k \to 0} \left( 2 \cdot \frac{\cos(k) - 1}{k} \right) \\
= 2 \cdot 0 = 0.
\]

Now the equation of the tangent line at \( x = c \) is

\[
y - f(c) = f'(c)(x - c),
\]
so, in this case

\[
y - 1 = 0 \cdot (x - 0),
\]
or

\[
y = 1.
\]
4) [15 points] Let

\[ \lim_{x \to 1} f(x) = 3, \quad \lim_{x \to 1} g(x) = -2, \quad \lim_{x \to 1} h(x) = +\infty. \]

Compute the following limits. [If a limit does not exist and is neither +\(\infty\) nor -\(\infty\), write DNE. You do not need to show work here.]

(a) \( \lim_{x \to 1} f(x) - g(x) = 3 - (-2) = 5 \)

(b) \( \lim_{x \to 1} g(x) \cdot h(x) = -\infty \)

(c) \( \lim_{x \to 1} f(x)/h(x) = 0 \)

(d) \( \lim_{x \to 1} h(x)/g(x) = -\infty \)

(e) \( \lim_{x \to 1} \arctan(x - h(x)) = -\pi/2 \)
5) [15 points] Give a [finite] closed interval in which we have a solution to $3^x = x^2$. [Justify!]

Solution. We use the Intermediate Value Theorem.
Let $f(x) = 3^x - x^2$. Then, [by trial an error], we have

$$f(-1) = 1/3 - 1 = -2/3 < 0$$

and

$$f(0) = 1 - 0 = 1 > 0.$$

Since $f(x)$ is continuous for all $x$ [and so, in particular on $[-1, 0]$], we have, by the Intermediate Value Theorem, that there is a $c \in [-1, 0]$ such that $f(c) = 0$, i.e., $3^c = c^2$.

In other words, the equation has a solution [namely $x = c$] in the closed interval $[-1, 0]$. □
6) [15 points] Let \( f(x) \) be a function for which \( f'(-2) = 2 \), \( f'(0) = 0 \) and \( f'(2) = -1 \). Mark the option below that could represent the graph of this function. [You do not have to justify, but in that case, we cannot give partial credit! If you do write an explanation for your choice, we can.]

(a)

(b)
(e) None of the above.

Solution. Only (b) and (c) have positive slope [i.e., increasing] at $x = -2$, slope zero [i.e., horizontal] at $x = 0$ and negative slope [i.e., decreasing] at $x = 2$. But (b) has a small [i.e., close to zero] slope at $x = -2$, so different from 2, and a very steep negative slope at $x = 2$, so different from $-1$. So, (c) is the correct answer.