This is a take-home exam: You cannot talk to anyone (except me) about anything about this exam and you can only look at our book (Walker), class notes and solutions to our HW problems posted by me. No other reference, including the Internet. Failing to follow these instructions will give you a zero in the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

Due date: 10am on Monday (12/09), which is the official time our final would end. If you cannot bring it to my office, a scanned/typed copy by e-mail would be OK.

1) Consider $D_6 = \{1, \rho, \rho^2, \ldots, \rho^5, \phi, \rho \phi, \rho^2 \phi, \ldots, \rho^5 \phi\}$. Show that $D_6/Z(D_6)$ is not Abelian.

[Hint: Remember that $Z(D_6) = \langle \rho^3 \rangle$.]

2) Let $G$ be a finite group with $|G| = n$ and $H \leq G$ with $(G : H) = k$.

(a) Prove that if $n$ does not divide $k!$, then there is $N \leq H$ with $N \neq \{1\}$ and $N \triangleleft G$.

[Hint: This is Problem 2.4.13.]

(b) Prove that if $k = p$, where $p$ is the least prime dividing $n$, then $H \triangleleft G$. [Here $n$ might be divisible by $p$ more than once. In particular, $n$ could be a power of $p$.] [Hint: This is related to the previous part! You can use it even if you could not do it.]

3) Let $G$ be a group of order 190. Prove that is contains a normal subgroup of order 95. Moreover, prove that this subgroup is cyclic.

4) Let $R$ be a ring with 1 [and 1 $\neq 0$], $M$ be an ideal of $R$ and suppose that $R^\times = R \setminus M$. [Remember, $R \setminus M \overset{\text{def}}{=} \{x \in R : x \notin M\}$] Prove that $M$ is a maximal ideal and that there are no other maximal ideals besides $M$.

5) Let $R$ be a commutative ring with 1 [and 1 $\neq 0$] and $I$ be an ideal of $R$. Let $\text{rad}(I) = \{x \in R : x^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}$.

(a) Prove that $\text{rad}(I)$ is an ideal of $R$ containing $I$. [Hint: Newton’s formula:

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

works in commutative rings! [No need to prove it.]]

(b) Prove that if $P$ is a prime ideal with $I \subseteq P$, then $\text{rad}(I) \subseteq P$. 