1) [20 points] Consider the following permutations in $S_7$:

\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 7 & 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \tau = (1, 7, 4)(1, 3, 5)(2, 6) \quad [\text{note it’s not disjoint!}] \]

[No need to show work for the items below!]

(a) Write $\sigma \cdot \tau$ in the matrix representation [as $\sigma$ was given].

\[ \sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 4 & 2 & 6 & 3 & 7 \end{pmatrix}. \]

(b) Write $\sigma$ as a product of disjoint cycles.

\[ \text{Solution.} \quad \sigma = (1, 2, 3)(4, 7, 6, 5). \]

(c) What is $|\sigma|$?

\[ \text{Solution.} \quad |\sigma| = \text{lcm}(3, 4) = 12. \]

(d) Write $\sigma$ as a product of transpositions.

\[ \text{Solution.} \quad \sigma = (1, 3)(1, 2)(4, 5)(4, 6)(4, 7). \]

(e) Find $\rho$ such that $\rho \tau \rho^{-1} = (2, 7, 5)(2, 3, 1)(4, 6)$. If there is no such $\rho$, say so and justify.

\[ \text{Solution.} \quad \text{We have } \rho(1) = 2, \rho(7) = 7, \rho(4) = 5, \rho(1) = 2, \rho(3) = 3, \rho(5) = 1, \rho(2) = 4 \quad \text{and} \quad \rho(6) = 6. \quad \text{So:} \]

\[ \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 5 & 1 & 6 & 7 \end{pmatrix}. \]
2) [20 points] Consider \( D_6 = \{1, \rho, \rho^2, \ldots, \rho^5, \phi, \rho\phi, \rho^2\phi, \ldots, \rho^5\phi\} \) and its subgroup \( H \overset{\text{def}}{=} \langle \rho^3, \phi \rangle \).

(a) Compute \( (\rho^3\phi)^{231} \cdot (\rho^4\phi)^{-1} \cdot \rho^{601} \). [Your answer should be one of the listed elements above: \( \rho^i \) or \( \rho^i\phi \), with \( i \in \{0, \ldots, 5\} \).]

\[
(\rho^3\phi)^{231} \cdot (\rho^4\phi)^{-1} \cdot \rho^{601} = (\rho^3\phi) \cdot (\rho^4\phi) \cdot \rho = \rho^3(\phi\rho^4)\phi\rho = \rho^3(\rho^2\phi)\phi\rho = \rho^5\phi^2\rho = \rho^5\rho = \rho^6 = 1
\]

(b) List all the elements of \( H \). [No need to justify or show work.]

\[
H = \{1, \rho^3, \phi, \rho^3\phi\}.
\]

(c) Is \( H \triangleleft D_6 \)? [Justify!]

\[
\text{Solution. No, as } \phi \in H, \text{ but } \rho\phi\rho^{-1} = \rho^2\phi \not\in H.
\]
3) [15 points] Show that $A_5 \ncong D_{30}$. [Here, it suffices to give a *structural* property that one of the groups has, but the other does not.]

Proof. We have that $A_5$ is simple [i.e., the only normal subgroups are $\{1\}$ and the groups itself], but $D_{30}$ is not. For instance, $|\langle \rho \rangle| = 30$, so it has index 2 in $D_{30}$ and hence it is a proper normal subgroups different from $\{1\}$. [Or, $Z(D_{30}) = \{1, \rho^{15}\}$ is another example of a normal subgroups different from $\{1\}$.

4) [20 points] Let $N \triangleleft G$ and $\phi \in \text{Aut}(G)$. Show that $\phi(N) \triangleleft G$.

Proof. Let $y \in G$ and $m \in \phi(N)$. [We need to show that $ymy^{-1} \in \phi(N)$.] Since $\phi$ is a bijection [and hence onto], there is $x \in G$ such that $\phi(x) = y$. Also, by definition [of $\phi(N)$], there is $n \in N$ such that $\phi(n) = m$.

Then:

$$ymy^{-1} = \phi(x)\phi(n)\phi(x)^{-1} = \phi(x)\phi(n)\phi(x^{-1}) = \phi(xnx^{-1}).$$

Since $N \triangleleft G$, we have that $xnx^{-1} \in N$, and hence, $\phi(xnx^{-1}) = ymy^{-1} \in \phi(N)$. 

□
5) In this problem, we will prove that if \( p \neq 2 \) is a prime and \( G \) is a group with \( |G| = 2p \), then \( G \) has a normal subgroup of order \( p \). [It is also true for \( p = 2 \) and it can be done directly. But here we will assume that \( p \neq 2 \).] **You can use a previous item even if you haven’t proved it!**

(a) [10 points] Assume that there is no subgroup of order \( p \). Prove that \( G \) is then Abelian. **[Hint: Use an old HW problem.]**

**Proof.** If \( G \) has an element of order \( p \), say \( x \), then it has a subgroup of order \( p \), namely \( \langle x \rangle \). So, it cannot have such element. Therefore, by Lagrange, every element has order 2, 2 or 1.

If \( |x| = 2p \), then \( |x^2| = |x|/(2,|x|) = p/(2,p) = p \), which is a contradiction. So, no element has order 2, and hence every element has order 2 or 1.

Thus, for all \( x \in G \), we have that \( x^2 = 1 \). As seen in a previous HW problem, this means that \( G \) is Abelian. 

(b) [10 points] Still assuming that there is no subgroup of order \( p \), show that \( G \) has a subgroup, say \( N \), of order 2. Since \( G \) is Abelian (by (a)), we have that \( N \triangleleft G \). Derive a contradiction by looking at \( G/N \).

**Proof.** Since every element has order 2 or 1 and only the identity has order one, we have that for any \( x \in G \setminus \{1\} \), \( N \overset{\text{def}}{=} \langle x \rangle \) has order 2.

Since \( |N| = 2 \), we have that \( |G/N| = p \). So, an element \( yN \in G/N \setminus \{1N\} \) has order \( p \) [as \( p \) is prime]. But, \( y \) must have order 2, as seen above. So, \( (yN)^2 = y^2N = N \), a contradiction since \( |yN| = p > 2 \).

(c) [5 points] So, from the previous items, there is a subgroup of \( G \), say \( H \), of order \( p \). Prove that \( H \triangleleft G \).

**Proof.** Since \( |G| = 2p \) and \( |H| = p \), we have that \( (G : H) = |G|/|H| = 2 \), and hence it is normal. 

**Proof.**