Here are some algebraic structures we will study this *year*:

- Rings
- Fields
- Groups
- Vector Spaces (in more details)
- Modules
Rings

Perhaps the most familiar algebraic structure is *rings*.

Rings have two operations: sum and product. [Product here is not *scalar* product, but product between two elements!] Of course, we ask these operations to satisfy some common properties: associativity, distributive, commutativity [sometimes], etc.

Examples are:

- \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \);
- \( R[X] \) [polynomials with variable \( X \) and coefficients in \( R \)] where \( R \) is one of the examples above [or a *commutative* ring]
- \( M_n(R) \) [\( n \times n \) matrices with entries in \( R \)] where \( R \) is a one of the examples above [or a *commutative* ring]

On the other hand, \( \mathbb{N} \) is not a ring, as it lacks “negatives” of elements.
Commutative Rings

Note that in the last example [matrices] the \textit{multiplication} is not commutative! We require the addition to \textit{always} be commutative, but not multiplication. When multiplication is commutative, we call the ring a \textit{commutative ring}.
Another important example is $\mathbb{Z}/n\mathbb{Z}$: integers modulo $n$. [This is an example from Math 351.] Remember that in $\mathbb{Z}/n\mathbb{Z}$, you perform operations [sum and product] just as in $\mathbb{Z}$, but identify:

\[
\begin{align*}
\cdots &= -2n = -n = 0 = n = 2n = \cdots \\
\cdots &= -2n + 1 = -n + 1 = 1 = n + 1 = 2n + 1 = \cdots \\
\cdots &= -2n + 2 = -n + 2 = 2 = n + 2 = 2n + 2 = \cdots \\
&\vdots \\
\cdots &= -n - 1 = -1 = n - 1 = 2n - 1 = 3n - 1 = \cdots
\end{align*}
\]

[Ex: In $\mathbb{Z}/4\mathbb{Z}$, $3 + 3 = 6 = 2$ and $3 \cdot 3 = 9 = 1$.]

Note that $\mathbb{Z}/n\mathbb{Z}$ has $n$ elements.
Another familiar algebraic structure is \textit{fields}.

Basically fields are commutative rings [“with 1”] for which every non-zero element has an \textit{inverse}: if \( a \neq 0 \), then there is \( b \) [also in the field] such that \( ab = 1 \). [So, we can “divide” by non-zero elements.]

Examples are \( \mathbb{Q} \), \( \mathbb{R} \), \( \mathbb{C} \). [Note that \( \mathbb{Z} \), \( R[X] \), \( M_n(R) \) are \textit{not} fields.]

Another example is \( F(X) \), which is the set of all \textit{rational functions} [i.e., quotient of polynomials, with non-zero denominator] with coefficients in some field \( F \).

Finally \( \mathbb{Z}/p\mathbb{Z} \) is a field if [and only if] \( p \) is \textit{prime}. 
**Vector Spaces** are the structures studied on Math 251: a set with two operations, sum and *scalar* multiplication. [You multiply an element of the vector space by a scalar, *not* by another element of the vector space.]

In Math 251 scalars were real numbers, but more generally scalars can be elements of any *field* [as above], such as \( \mathbb{C} \), \( \mathbb{Q} \), \( \mathbb{Z}/7\mathbb{Z} \), etc.

In Math 251 you’ve seen diagonalization of matrices. You’ve seen that it is not always possible! One of the main topics will be to find out the “next best thing(s)”: *rational and Jordan canonical forms*. 
Modules are like vector spaces, except the “scalars” are not in a field, but in a *ring*. This makes things *much* more complicated, especially if the ring is non-commutative.

We will deal only very briefly with modules and only over [a special case of] commutative rings. We will only deal with them because they give a “natural” way to prove of the canonical forms [for vector spaces] results.
Algebras are modules [or vector spaces] which are also rings. Thus, we have sum and both multiplication and scalar multiplication.

The main examples are:

- $R[X]$ [polynomials with coefficients in $R$ and variable $X$];
- $M_n(R)$ [$n \times n$ matrices with entries in $R$];

where $R$ is a commutative ring [and the scalars are the elements of $R$].

We will likely not deal with algebras [at least explicitly] in this course.