What’s Calculus?

**Answer:** Next semester! (Fundamental Theorem of Calculus, by Newton and Leibniz.)

*Virtually all of modern science uses calculus!* Physics, engineering, statistics, biology (modeling), etc.

**This semester:** *Differential* Calculus. (Tangent lines.)
**Next semester:** *Integral* Calculus. (Areas.)
Computing *areas* is one the most classical problems in mathematics. (The term *geometry* comes from the Greek “*land (or earth) measurement*”.) The idea is to compare the space taken by a plane shape with the space taken by one square of side 1.

- **Area of rectangle:** length of base times length of height.
- **Area of triangle:** half of the length of base times length of height. (From this, we can get areas of polygons.)
- **Area of Circle:** $\pi$ times the square of the radius. *Why????* How did one find that out?
How about the area of an ellipse? Say \((x/3)^2 + (y/2)^2 = 1\)?

What’s its area?
Other Areas

How about the area between a line and a parabola? How about two parabolas?

These are hard questions! Answers in Math 142.
Suppose that you know that a particle in moving along a straight line such that $t$ seconds after we start observing the movement, the position of the particle is $t^2$ meters from the original position. In other words, the position of the particle is given by the function $s(t) = t^2$.

One can clearly see that the particle is accelerating.
Average Speed

Since we know the position at any time, we should be able to find out *everything* about the movements of the particle! (Not only its position at a given time.)

For instance, we can find the **average speed** of the particle in a period. For instance, the average speed between \( t = 1 \) and \( t = 2 \):

\[
\frac{\Delta s}{\Delta t} = \frac{s(2) - s(1)}{2 - 1} = \frac{4 - 1}{2 - 1} = 3.
\]

Between \( t = 2 \) and \( t = 3 \):

\[
\frac{\Delta s}{\Delta t} = \frac{s(3) - s(2)}{3 - 2} = \frac{9 - 4}{3 - 2} = 5.
\]
Instantaneous Speed

But how about the **instantaneous speed** at, say \( t = 2 \). (What is the speedometer showing if we look at it at \( t = 2 \)?) *Much harder!*

**Idea:** I might not be able to know the *exact* speed, but I can get a very good idea: find the *average* speed of a *tiny* interval starting at \( t = 2 \). The smaller the interval is, the less time the particle had to change its speed, so the closest we get to the real speed at \( t = 2 \)!

So, we find the average speed between \( t = 2 \) and \( t = 2 + \Delta t \), for \( \Delta t \) *small*. Here are some computations:

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>Aver. Sp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.1</td>
</tr>
<tr>
<td>0.01</td>
<td>4.01</td>
</tr>
<tr>
<td>0.001</td>
<td>4.001</td>
</tr>
<tr>
<td>0.0001</td>
<td>4.0001</td>
</tr>
</tbody>
</table>

So, the speed at \( t = 2 \) is pretty close to 4.0001. (Is it 4?)
Computing Average Speeds

The computations done for the average speed on the previous slide can be done quite quickly by a computer (or even calculator). But imagine for a second we have to compute *lots* of different average speeds *by hand*! Here is a smart way of doing it: *find a formula for the average speed!* The average speed between $t = t_0$ and $t = t_0 + \Delta t$ is:

$$\frac{\Delta s}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{(t_0 + \Delta t) - t_0} = \frac{(t_0 + \Delta t)^2 - t_0^2}{\Delta t}$$

$$= \frac{(t_0^2 + 2t_0\Delta t + (\Delta t)^2) - t_0^2}{\Delta t}$$

$$= \frac{2t_0\Delta t + (\Delta t)^2}{\Delta t} = \frac{2t_0\Delta t + (\Delta t)^2}{\Delta t}$$

$$= 2t_0 + \Delta t.$$
Computing Average Speeds

This makes it easy to compute average speeds and estimate instantaneous speeds:

<table>
<thead>
<tr>
<th>$t_0$</th>
<th>$\Delta t$</th>
<th>Aver. Sp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2.5</td>
<td>0.01</td>
<td>5.01</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>6.01</td>
</tr>
<tr>
<td>4</td>
<td>0.01</td>
<td>8.01</td>
</tr>
</tbody>
</table>

In particular, the *instantaneous* speed at $t = 2.5$ is approximately 5.01, the *instantaneous* speed at $t = 3$ is approximately 6.01, the *instantaneous* speed at $t = 4$ is approximately 8.01.
Instantaneous Speed

But how do we find the exact instantaneous speed? The idea is that we want \( \Delta t = 0 \). But this doesn’t seem to make sense:

\[
\frac{\Delta s}{\Delta t} = \frac{s(t_0 + 0) - s(t_0)}{t_0 + 0 - t_0} = \frac{s(t_0) - s(t_0)}{t_0 - t_0} = \frac{0}{0}!
\]

But we cannot divide by 0!

On the other hand, we have a formula for the average speed \( \Delta s/\Delta t: 2t_0 + \Delta t \). So, here, we can make \( \Delta t = 0 \) without dividing by 0! Hence, the instantaneous speed of the particle at \( t = t_0 \) is \( 2t_0 \).

So, the (instantaneous) speed at \( t = 2 \) is 4, the (instantaneous) speed at \( t = 3 \) is 6, the (instantaneous) speed at \( t = 4 \) is 8, etc.
Geometrical Interpretation of Average Speed

Now let’s look at the geometry of the average speed. The formula \( \frac{\Delta s}{\Delta t} \) is basically a slope \( \left( \frac{\Delta y}{\Delta x} \right) \). The average speed between \( t = t_0 \) and \( t = t_0 + \Delta t \) is the slope of the line secant to the graph of \( s(t) \) through \( t = t_0 \) and \( t = t_0 + \Delta t \).
So, what is the geometrical interpretation of the *instantaneous* speed? It is the slope of the **tangent line** at \( t = t_0 \)!

\[ \Delta t = 0; \text{ slope} = 2.0 \]
Tangent Line

The **tangent line** is geometrically defined precisely as in the previous pictures: take secant lines and make the second point go approach the first.

Another way to see it: if a curve is smooth (no sharp edge), by *zooming in enough*, it starts to look like a straight line. This straight line is the tangent line! (A line which is not tangent makes an angle!)

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size: 0.002 × 0.001
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![Graph showing tangent, secant, and original function](image-url)
A bit of terminology: the **rate of change** of the position of a particle is its *instantaneous* speed. More precisely, it is how fast the position change when the time changes.

As we’ve seen, the speed (i.e., rate of change) of the position $s(t)$ at $t = t_0$ is the *slope of the tangent line to the graph $s = s(t)$ at $t = t_0$*. 

In general, the rate of change of a function $f(x)$ at $x = x_0$ is the *slope* of the tangent line to the graph $y = f(x)$ at $x = x_0$. This tells us how fast is the $y$ value changing at $x = x_0$. 
As we’ve seen, to compute rates of change (or slopes of the tangent line) of \( y = f(x) \) at \( x = 0 \), we do:

- Consider the ratio: \( \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \);
- simplify so that we don’t have a \( \Delta x \) in the denominator;
- replace \( \Delta x \) by 0.
Example

Consider \( f(x) = x^2 - x \). What is the slope of the tangent line at \( x = 1 \)?

\[
\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{[(1 + \Delta x)^2 - (1 + \Delta x)] - (1^2 - 1)}{\Delta x} \\
= \frac{[(1^2 + 2\Delta x + (\Delta x)^2) - (1 + \Delta x)] - 0}{\Delta x} \\
= \frac{\Delta x + (\Delta x)^2}{\Delta x} \\
= 1 + \Delta x.
\]

Now, we can make \( \Delta x = 0 \) in the above expression, obtaining the answer: 1.
Problems

There are two problems: the above is not *mathematically precise*! It is a procedure, but does not *define* rate of change/slope of tangent line precisely.

Also, does the procedure always work? Consider the tangent line of $y = \sin(x)$ at $x = 0$. How do you simplify

$$\frac{\sin(0 + \Delta x) - \sin(0)}{\Delta x} = \frac{\sin(\Delta x)}{\Delta x}$$

to cancel out the $\Delta x$ in the denominator (so that we can replace it by 0)? *Hard!*

We need the notion of *limit*!