1) What’s the coefficient of $x^{20}$ in $(2 + 3x^4)^{100}$? [You do not need to evaluate powers and binomials.]

Solution. We have

$$(2 + 3x^4)^{100} = \sum_{i=0}^{100} \binom{100}{i} (3x^4)^i 2^{100-i}.$$ 

Hence, the coefficient is $\binom{100}{5} 3^5 2^{95}$ [i.e., we take $i = 5$].

\[\square\]
2) [Remember: if $a, b \in \mathbb{Z}$, then $a$ divides $b$ if there exists $q \in \mathbb{Z}$ such that $b = a \cdot q$.] Let $a, b, d \in \mathbb{Z}$. Prove that $d$ divides $a$ and $b$ if, and only if, $d$ divides $a$ and $a + b$.

Proof. [$\Rightarrow$] Suppose that $d$ divides $a$ and $b$. [We need to show that $d$ divides $a$ and $a + b$.] Then, by definition there $q_1, q_2 \in \mathbb{Z}$ such that $a = q_1 \cdot d$ and $b = q_2 \cdot d$. Then, $a + b = q_1 \cdot d + q_2 \cdot d = (q_1 + q_2) \cdot d$, and hence [since $\mathbb{Z}$ is closed under addition] $d$ divides $a + b$ by definition [of division]. Since $d$ also divides $a$ [by assumption], we have that $d$ divides $a$ and $a + b$.

[$\Leftarrow$] Suppose now that $d$ divides $a$ and $a + b$. [We need to show that $d$ divides $a$ and $b$.] Then, by definition, there are $q_1, q_3 \in \mathbb{Z}$ such that $a = q_1 \cdot d$ and $a + b = q_3 \cdot d$. Hence, $b = (a + b) - b = q_3 \cdot d - q_1 \cdot d = (q_3 - q_1) \cdot d$, and thus [since $\mathbb{Z}$ is closed under subtraction] we have that $d$ divides $b$ by definition [of division]. Since $d$ also divides $a$ [by assumption], we have that $d$ divides $a$ and $b$. 

$\square$
3) Prove or disprove: $A \setminus (B \cap C) = (A \setminus C) \cup (C \setminus B)$.

Solution. The statement is false! [Again, it suffices to give a counterexample.] Let $A = B = \emptyset$ and $C = \{1\}$. Then, $A \setminus (B \cap C) = \emptyset$. Also, $A \setminus C = \emptyset$ and $C \setminus B = \{1\}$. Hence $(A \setminus C) \cup (C \setminus B) = \{1\} \neq \emptyset = A \setminus (B \cap C)$. □
4) Let \( \mathcal{R} \) be the relation on \( \mathbb{R} \) given by \( a \mathcal{R} b \iff a - b \in \mathbb{Z} \).

(a) Prove that \( \mathcal{R} \) is an equivalence relation.

\[
\text{Proof.} \quad [\text{Reflexive:}] \quad \text{[We need to prove that } x \mathcal{R} x \text{ for all } x \in \mathbb{R}.] \quad \text{Given } x \in \mathbb{R}, \text{ we have that } x - x = 0 \in \mathbb{Z}. \text{ Thus, } x \mathcal{R} x \text{ [by definition].}
\]

\[
[\text{Symmetric:}] \quad \text{Suppose that } x \mathcal{R} y. \quad [\text{We need to prove that } y \mathcal{R} x.] \quad \text{Then, [by definition] we have that } x - y \in \mathbb{Z}. \text{ Thus, } -(x - y) = y - x \in \mathbb{Z} \text{ [as } 0 \in \mathbb{Z} \text{ and } \mathbb{Z} \text{ is closed under subtraction]. Hence, } y \mathcal{R} x \text{ [by definition].}
\]

\[
[\text{Transitive:}] \quad \text{Suppose that } x \mathcal{R} y \text{ and } y \mathcal{R} z. \quad [\text{We need to prove that } x \mathcal{R} z.] \quad \text{By definition, we have that } x - y, y - z \in \mathbb{Z}. \text{ Hence, [since } \mathbb{Z} \text{ is closed under addition] we have that } (x - y) + (y - z) = x - z \in \mathbb{Z}, \text{ and thus } x \mathcal{R} z \text{ [by definition].}
\]

\( \square \)

(b) Give three elements in the equivalence class \( 0.312 \), at least one of which is negative, and three elements not in \( 0.312 \), at least one of which is negative. [No need to justify this part.]

\[
\text{Solution.} \quad \text{We have that } 0.312, 1.312, \underbrace{0.312 - 1}_{= -0.688} \in 0.312, \text{ and } -1, 0, 1 \notin 0.312. \quad \square
\]
5) Find a closed formula for the recursion \( a_0 = 0, \ a_n = 2 \cdot a_{n-1} - 3 \) for \( n \geq 1 \). [You don’t have to show me how you came up with the formula, but you have to prove that it is correct.]

**Solution.** We have

\[
\begin{align*}
a_0 &= 0 \\
a_1 &= -3 \\
a_2 &= 2 \cdot (-3) + (-3) \\
a_3 &= 4 \cdot (-3) + 2 \cdot (-3) + (-3) \\
a_4 &= 8 \cdot (-3) + 4 \cdot (-3) + 2 \cdot (-3) + (-3) \\
&\vdots \\
a_n &= 2^{n-1} \cdot (-3) + 2^{n-2} \cdot (-3) + \cdots + 2^1 \cdot (-3) + 2^0 \cdot (-3) \\
&= (-3) \cdot \left( 2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 \right) \\
&= -3 \cdot \frac{2^n - 1}{2 - 1} = -3 \cdot (2^n - 1).
\end{align*}
\]

So, we claim that \( a_n = -3 \cdot (2^n - 1) \), and prove it by induction.

For \( n = 0 \), we have that \( a_0 = 0 = -3 \cdot (2^0 - 1) \).

Now, suppose that \( a_n = -3 \cdot (2^n - 1) \). [We need to prove that \( a_{n+1} = -3 \cdot (2^{n+1} - 1) \).]

We then have:

\[
\begin{align*}
a_{n+1} &= 2 \cdot a_n - 3 & [\text{recurrence}] \\
&= 2 \cdot (-3 \cdot (2^n - 1)) - 3 & [\text{ind. hyp.}] \\
&= -3 \cdot (2 \cdot (2^n - 1) + 1) & [\text{factor } -3] \\
&= -3 \cdot (2^{n+1} - 2 + 1) = -3 \cdot (2^{n+1} - 1).
\end{align*}
\]

\(\square\)
6) Let \( f : X \to Y \) and \( A \subseteq Y \).

(a) Prove that if \( f \) is onto, then \( f(f^{-1}(A)) = A \).

\[ \text{Proof. (\subseteq \text{ Let } y \in f(f^{-1}(A)). \text{ [We need to show that } y \in A.] \text{ Then, by definition of direct image, there exists } x \in f^{-1}(A) \text{ such that } y = f(x). \text{ But, by definition of preimage, we have that } x \in f^{-1}(A) \text{ means that } f(x) \in A. \text{ Since } y = f(x), \text{ we have that } y \in A. \text{ [Note that we did not use the fact that } f \text{ is onto here.]} \right] \]

\[ \text{\text{\supseteq \text{ Let } y \in A. \text{ [We need to show that } y \in f(f^{-1}(A)).]\text{ Since } f \text{ is onto, there exists } x \in X \text{ such that } y = f(x). \text{ Since } y \in A, \text{ by definition of preimage, we have that } x \in f^{-1}(A). \text{ Since } y = f(x) \text{ and } x \in f^{-1}(A), \text{ by definition of direct image, } y \in f(f^{-1}(A)). \text{ [Note that the fact } f \text{ is onto is used in this part.]} \]}

(b) Give an example of \( f \) and \( A \) such that \( f(f^{-1}(A)) \neq A \).

\[ \text{Solution. Let } f : \mathbb{R} \to \mathbb{R} \text{ be the function } f(x) = x^2, \text{ and take } A = \{-1, 1\}. \text{ Then, } f^{-1}(A) = \{-1, 1\}, \text{ and thus } f(f^{-1}(A)) = \{1\} \neq \{-1, 1\}. \]
7) Prove by induction that \( \frac{n}{n + 1} \geq \frac{1}{2} \) for all \( n \in \mathbb{N} \). You can use any property of inequalities we’ve seen before, as long as you state it clearly!

[Hint: Prove first that \((n + 1)^2 > n(n + 2)\). [You do not need induction for that!] Then, note that \( \frac{n + 1}{n + 2} = \frac{n}{n + 1} \cdot \frac{(n + 1)^2}{n(n + 2)} \).]

Proof. First, observe that \((n + 1)^2 = n^2 + 2n + 1 > n^2 + 2n = n(n + 2)\). Then, for \( n \neq -1, -2 \), we have that \( \frac{(n + 1)^2}{n(n + 2)} > 1 \).

Now, we prove the statement by induction. For \( n = 1 \), we have \( 1/(1 + 1) \geq 1/2 \).

Suppose then that \( \frac{n}{n + 1} \geq \frac{1}{2} \) for some \( n \geq 1 \). [We need to prove that \( \frac{n + 1}{n + 2} \geq \frac{1}{2} \).] As observed above, since \( n \neq -1, -2 \), we have that \( \frac{(n + 1)^2}{n(n + 2)} > 1 \), and then

\[
\frac{n + 1}{n + 2} = \frac{n}{n + 1} \cdot \frac{(n + 1)^2}{n(n + 2)} \geq \frac{1}{2} \cdot 1 = \frac{1}{2}.
\]

[Here, we’ve used the fact that if \( 0 < a \leq b \) and \( 0 < c \leq d \), then \( ac \leq bd \).]
8) Suppose that $a$ and $b$ are elements of an ordered field [you can think of $\mathbb{R}$ if you want] that have $n$-th roots, and $0 < a < b$. Prove that for all $n \in \mathbb{N}$ we have that $a^{1/n} < b^{1/n}$. [This is straight from your HW! You can use anything we’ve proved in class or HW about inequalities with integer exponents, as long as you state it clearly!]

**Proof.** We prove the result by contradiction. Suppose that $a^{1/n} \geq b^{1/n}$. [We must derive a contradiction.]

If $a^{1/n} = b^{1/n}$, then $a = (a^{1/n})^n = (b^{1/n})^n = b$, which is a contradiction [as $a < b$].

If $a^{1/n} > b^{1/n}$, since we know $b^{1/n} > 0$ [by definition of $n$-th root], we have that $a = (a^{1/n})^n > (b^{1/n})^n = b$ [as if $0 < x < y$, then $x^n < y^n$ for all $n \in \mathbb{N}$], which again contradicts $a < b$.

\[ \square \]
9) Let $F$ be a field. [Remember that if $a \in F$, then $n(a) = n(1) \cdot n(a)$, $n(n(a)) = a$, and if $a, b \in F \setminus \{0\}$, then $q(a \cdot b) = q(a) \cdot q(b)$. You can use those, without proving them, in both parts below.]

(a) Prove that $q(n(1)) = n(1)$. [Hint: Use that if $x \cdot a = 1$, then $x = q(a)$.]

Proof. We have that $n(1) \cdot n(1) = n(n(1)) = 1$. As stated in the hint, this means that $n(1) = q(n(1))$. \hfill \qed

(b) Prove that if $a \in F \setminus \{0\}$, then $q(n(a)) = n(q(a))$. [Hint: It might help to use (a).]

Proof. We have

$q(n(a)) = q(n(1) \cdot a) = q(n(1)) \cdot q(a) = n(1) \cdot q(a) = n(q(a))$. \hfill \qed