1) [20 points] Mark true or false. Justify your answers only for the ones which are false. [No need to justify if true.]

(a) \( P = \{\{1, 2, 3, 4, 5\}\} \) is a partition of \( \{1, 2, 3, 4, 5\} \).

\textbf{Answer:} T.

(b) \( P = \{\{1\}, \{2, 5\}, \{3, 4\}\} \) is a partition of \( \{1, 2, 3, 4, 5, 6\} \).

\textbf{Answer:} F, as 6 is not in any of the sets in the partition.

(c) Let \( X \) be the set of all people in the world, and define the relation \( R \) by \( xRy \) if \( x \) is married to \( y \). Then, \( R \) is an equivalence relation.

\textbf{Answer:} F, as a person who is married is not married to herself/himself.

(d) Let \( X \) be the set of all people in the world, and define the relation \( R \) by \( xRy \) if \( x \) and \( y \) have the same mother. Then, \( R \) is an equivalence relation.

\textbf{Answer:} T.

(e) The converse of an “if-then” statement is logically equivalent to the original statement [i.e., either both are false or both are true].

\textbf{Answer:} F, for example “if \( x = 1 \), then \( x > 0 \)” is true, but the converse “if \( x > 0 \), then \( x = 1 \)” is false.
2) [20 points] Give the completely simplified negation of the following statement. [Your answers should have no “nots” in them.]

For all $\epsilon > 0$, there is $\delta > 0$ such that either $|x| \geq \delta$ or $|x^2| < \epsilon$.

[Hint: Negate one part at a time, as it makes it easier to get partial credit.]

Solution.

$\neg (\text{For all } \epsilon > 0, \text{ there is } \delta > 0 \text{ such that either } |x| \geq \delta \text{ or } |x^2| < \epsilon.) \sim$

There exists $\epsilon > 0$ such that $\neg (\text{there is } \delta > 0 \text{ such that either } |x| \geq \delta \text{ or } |x^2| < \epsilon.) \sim$

There exists $\epsilon > 0$ such that for all $\delta > 0$, $\neg (\text{either } |x| \geq \delta \text{ or } |x^2| < \epsilon.) \sim$

There exists $\epsilon > 0$ such that for all $\delta > 0$, $|x| < \delta$ and $|x^2| \geq \epsilon$.

3) [20 points] Prove that if $A$ and $B$ are symmetric subsets of $\mathbb{R}$, then so is $A \cap B$.

Proof. As seen in class, a set $X$ is symmetric if, and only if, for all $x \in X$, we have that $-x \in X$.

So, let $x \in A \cap B$. Then, by definition of intersection, $x \in A$ and $x \in B$. Since $A$ and $B$ are symmetric, we have that $-x \in A$ and $-x \in B$. Hence, by definition of intersection again, we have that $-x \in A \cap B$. Thus, $A \cap B$ is symmetric. 


4) [20 points] Let $\mathcal{R}$ be the relation on $\mathbb{R}^2$ given by $(x_1, y_1) \mathcal{R} (x_2, y_2)$ if there exists $r \in \mathbb{R}$ such that $(x_2, y_2) = (x_1 + r, y_1 + r)$.

[So, be careful here! You will use ordered pairs for elements! I.e., use “assume $(x_1, y_1) \mathcal{R} (x_2, y_2)$” instead of “assume $x \mathcal{R} y$” in your proofs.]

(a) Prove that $\mathcal{R}$ is an equivalence relation.

Proof. [Reflexive:] Given $(x, y) \in \mathbb{R}^2$, we have that $(x, y) = (x + 0, y + 0)$, and hence $(x, y) \mathcal{R} (x, y)$ [since $0 \in \mathbb{R}$].

[Symmetric:] Suppose that $(x_1, y_1) \mathcal{R} (x_2, y_2)$. Then, there is $r \in \mathbb{R}$ such that $(x_2, y_2) = (x_1 + r, y_1 + r)$. Thus, $(x_1, y_1) = (x_2 - r, y_2 - r)$. Since $-r \in \mathbb{R}$ [as $-1 \in \mathbb{R}$ and $\mathbb{R}$ is closed under multiplication], we have that $(x_2, y_2) \mathcal{R} (x_1, y_1)$.

[Transitive:] Suppose that $(x_1, y_1) \mathcal{R} (x_2, y_2)$ and $(x_2, y_2) \mathcal{R} (x_3, y_3)$. Then, by definition, there are $r, s \in \mathbb{R}$ such that $(x_2, y_2) = (x_1 + r, y_1 + r)$ and $(x_3, y_3) = (x_2 + s, y_2 + s)$. Hence, $(x_3, y_3) = (x_1 + (r + s), y_1 + (r + s))$, and since $r, s \in \mathbb{R}$ and $\mathbb{R}$ is closed under addition, we have that $(x_1, y_1) \mathcal{R} (x_3, y_3)$.

(b) Give the equivalence class of $(0, 0)$.

Solution. We have:

$$\overline{(0, 0)} = \{(x, y) \in \mathbb{R}^2 : (0, 0) \mathcal{R} (x, y)\}$$

$$= \{(x, y) \in \mathbb{R}^2 : \exists r \in \mathbb{R} \text{ such that } (x, y) = (0 + r, 0 + r)\}$$

$$= \{(r, r) : r \in \mathbb{R}\}.$$

Hence, $\overline{(0, 0)}$ is the line $y = x$.  

\[ \square \]
5) [20 points]

(a) Prove that $x \not\in A \setminus C$ can be interpreted as saying that $x \not\in A$ or $x \in C$. [You need to give a formal proof! Wordy arguments will receive partial credit at best. **Hint:** Remember how to negate “and/or” statements.]

**Proof.** Observe that to say that $x \not\in A \setminus C$, is to negate $x \in A \setminus C$, i.e., to negate $(x \in A \text{ and } x \not\in C)$, which means that $x \not\in A$ or $x \in C$ by De Morgan’s Law.

[One can also use the contrapositive to prove this.]

(b) Let $A$, $B$ and $C$ be sets. Prove that $(B \cap C) \cup (B \setminus A) = B \setminus (A \setminus C)$. [**Hint:** Use item (a). Note that you can use it even if you did not do that part!]

**Proof.** [$\subseteq$:] Let $x \in (B \cap C) \cup (B \setminus A)$. Then, by definition of union, $x \in (B \cap C)$ or $x \in B \setminus A$. Then, by definition of intersection and complement, we have that $(x \in B \text{ and } x \in C)$ or $(x \in B \text{ and } x \not\in A)$.

Thus, we have that $x \in B$ and (either $x \in C$ or $x \not\in A$). Therefore, $x \in B$ and $x \not\in A \setminus C$ [from part (a)]. By definition of complement, $x \in B \setminus (A \setminus C)$.

[$\supseteq$:] Let $x \in B \setminus (A \setminus C)$. Then, $x \in B$ and $x \not\in A \setminus C$. This then means that $x \in B$ and (either $x \not\in A$ or $x \in C$) by part (a). So, either $(x \in B \text{ and } x \not\in A)$ or $(x \in B \text{ and } x \in C)$. Thus, $x \in B \setminus A$ or $x \in B \cap C$. Therefore, $x \in (B \cap C) \cup (B \setminus A)$.