We have:

\[ \sum_{t \in F_p} x(1-t^m) = \sum_{a \in F_p} N(a^m=a) \cdot x(1-a) = \sum_{a \in F_p} \left( \sum_{m \geq 2} \chi(a^m) \chi(1-a) \right) = \]

\[ = \sum_{a^m=1} \sum_{a \in F_p} \chi(a) \chi(1-a) = \sum_{a^m=1} \left( \sum_{a \in F_p} \chi(a) \chi(b) \right) = \]

\[ = \prod \left( 1 - \chi(a) \right) \]

Now since \( \chi \) is cyclic, \( \exists \lambda: \chi^m = \chi^\lambda \) with \( \lambda \in \{ \overline{1, m} \} \cap \{ \overline{1, p-1} \} \).

\[ | \sum_{a \in F_p} \chi(1-t^m) | \leq (d-1) \cdot p^{\frac{1}{2}} \]
(8.15) (I think we need $D \neq 0$, here.)

$$N = N(\gamma^2 - \chi + D) = \sum_{a - b = D} N(\gamma^2 = a) N(\chi = b) =$$

$$= \sum_{a - b = D} \left( \sum_{i=0}^{p-1} (a) \right) \left( \sum_{j=0}^{2} \chi(b) \right)$$

$$= \sum_{a - b = D} 1 + p(a) + \chi(b) + \chi^2(b) + p(a)\chi(b) + p(a)\chi^2(b) =$$

$$= p + \sum_{a=0}^{p-1} p(a) + \chi(b) + \chi^2(b) \sum_{a - b = D} \left( p(a)\chi(b) + p(a)\chi^2(b) \right)$$

assuming that $D \neq 0$

$$= p + \sum_{a+b=1} p(a)\chi(b) + p(a)\chi^2(b)$$

$$= p + \sum_{a+b=1} p(a)\chi(b)$$

$$= p + \pi + \pi \chi$$

Since $p(a) = \pm 1$ so $p = \overline{p}$.

Now, suppose that $x(2) = 1$ and $D = 1$. (i.e. $\pi = \text{J}(a, e)$)

Since $p = 1 \pmod{b}$, clearly $p = 1 \pmod{3}$. So, Problem 89

Suggest us that $a(x^3) = p(x(2) \cdot \text{J}(a, p)) = p \pi$
Now, we follow the idea of Proposition 8.3.4:

\[ g(x^3) = -1 \quad \text{(check the book at pg 96/97)} \]

\[ p \mid (x, p) \]

\[ p \omega = \frac{2}{\omega} \]

Now, since the range of \( x \) is in \( \mathbb{Z} \), and the range of \( p \) is in \( \{0, 4, 9\} \), \( \exists J(x, p) = a + bw, a, b \in \mathbb{Z} \)

We can repeat this whole argument with \( \bar{x} \) (as it also has order 3) and set \( p \mid J(x, p) = a + b \overline{\omega} \)

\[ g(x^3) \]

Thus:

\[ g(x^3) = g(x^3)^{1/2} = 1 \]

\[ a + bw = -1 \quad \text{(mod 3)} \]

\[ a + b \overline{\omega} = -1 \quad \text{(mod 3)} \]

Subtracting, we set \( 31 b \) (check pg 97 for some steps).

We hope that \( \sqrt{J(x, p)} \) is \( p \) (check at pg 94)

\[ a^2 - ab + b^2 = \]

\[ 4p = (2a - b)^2 + 3b^2 = A^2 + 27B^2 \quad \text{(A in unique when } A = 1 \text{ mod 3)} \]

\[ a = 3B \text{ as } 31b \]

Thus \( A = 1 \)

\[ N = p + \pi + \overline{\pi} = p + (a + bw) + (a + b \overline{\omega}) = p + \frac{2a - b}{A} \]

Ex: \( 31 = (-2)^2 + 27 \cdot 1^2 \)

\[ N = 31 + (-2) = 29 \] points!
\[ N(y^2 + x^4 = 1) = \sum_{a+b=1} N(y^2 = a) \cdot N(x^4 = b) = \]
\[ = \sum_{a+b=1} \left( \sum_{i=0}^{\frac{3}{2}} p_i(a) \right) \cdot \left( \frac{3}{2} x^i(b) \right) = \sum_{a+b=1} 1 + p(a) + x(b) + x^2(b) + p(a)x(b) + p(a)x^2(b) + p(a)x^3(b) = \]
\[ = p + J(x, x) + J(p, x^2) + J(p, x^3) = \]
\[ \begin{cases} x^3 = x & \text{if } x^3 = x \\ x^3 = x^2 & \text{if } x^3 = x^2 \end{cases} = \]
\[ = p + a + bi + J(p, p) + a - bi = \]
\[ \begin{cases} \text{order } x = 4 & \Rightarrow x^4 = p \\ \text{order } x = 2 & \Rightarrow x^2 = p \\ \text{order } x = 1 & \Rightarrow x = p \end{cases} \]
\[ = p + 2a - p(-1) = p + 2a - (-1)^{\frac{p-1}{2}} = p + 2a - 1 \]
\[ \text{as } p = p^{-1} \]
\[ \Rightarrow p \equiv 1 \pmod{4} \]
\( N(\gamma^2 = 1-x^4) = \sum_{x \in \mathbb{F}_p} N(\gamma^2 = 1-x^4) \)

\[
= \sum_{x \in \mathbb{F}_p} \left( \sum_{i=0}^{\frac{p-1}{4}} \binom{2m}{i} \gamma^i \right) = p + \sum_{x \in \mathbb{F}_p} \left( 1 + \binom{2m}{i} \right) \gamma^i \]

From (a) and (b):

\[
2a - 1 = \sum_{x \in \mathbb{F}_p} \binom{2m}{i} \gamma^i \]

Remember:

\[
p(\gamma) \equiv 2^{\frac{p+1}{2}} (\mod p)\]

\[
2a - 1 = \sum_{x \in \mathbb{F}_p^*} \gamma^{2m} \]

\[
= 1 + \sum_{x \in \mathbb{F}_p^*} \binom{2m}{i} \gamma^i \]

now, if \((p-1)k4i\), then \(\sum_{x \in \mathbb{F}_p^*} x^{4i} = 0\). (I'm pretty sure we did this, but could not find it). Set \(F_p^* = \langle \alpha \rangle\) Then \(\sum_{x \in \mathbb{F}_p^*} x^{4i} = \sum_{j=0}^{\frac{p-1}{4}} (\alpha^j)^{4i} = (\alpha^{4i})^{\frac{p-1}{4i}} = 0\).

If \(p \equiv 1 \pmod{4}\), then \(\sum_{x \in \mathbb{F}_p^*} x = p - 1 = -1\). Since the sum ranges from...
2a = 1 + \sum_{i=0}^{2m} \left( \frac{2m}{i} \right) (-1)^i \left( \frac{x^n x_i}{x} \right)