

# Solutions to (Selected) HW Problems

Math 300 – Fall 2008

November 27, 2008

*Please read:* I will *try* to post here a few solutions. The new solutions will be added to this same file. They might come with no explanation, just the “answer”. If yours do not match mine, you can try to figure out again. (Also, read the disclaimer below!) You can come to office hours if you want explanations for the unexplained answers. Be careful that just because our “answers” were the same, it doesn’t mean that you solved the problem correctly (it might have been a “fortunate” coincidence), and in the exams what matters is the solution itself. I will do my best to post somewhat detailed solutions, though.

*Disclaimer:* I will have to put these solutions together rather quickly, so they are subject to typos and conceptual mistakes. (I expect you to be a lot more careful when doing your HW than I when preparing these.) You can contact me if you think that there is something wrong and I will fix the file if you are correct.

**Note:** I will use the square brackets “[ ... ]” to include extra explanations and comments which are not quite necessary for the solution, but that should make things a bit more clear.

## Homework 1

### Pg. 8

1. We prove the contrapositive: “If  $x$  is rational, then  $x + p$  is rational.” [We are still assuming that  $p$  is rational.]

Since  $\mathbb{Q}$  is closed under addition, and  $x, p \in \mathbb{Q}$ , we must have that  $(x + p) \in \mathbb{Q}$ .

The statement also holds if we change  $x + p$  by  $x \cdot p$ , as  $\mathbb{Q}$  is also closed under multiplication. Just use the contrapositive again: “If  $x$  is rational, then  $x \cdot p$  is rational.”

Since  $\mathbb{Q}$  is closed under multiplication, and  $x, p \in \mathbb{Q}$ , we must have that  $(x \cdot p) \in \mathbb{Q}$ .

2. (a) There exists  $\epsilon > 0$  such that for all integers  $n$  we have  $1/n \geq \epsilon$ .  
(b) There exists a real number  $x$  such that for all integers  $n$  we have  $x \geq n$ .  
(c) There exist irrationals  $x$  and  $y$  such that the sum  $x + y$  is rational.
3. (a) *Hypotheses:*  $x$  is a real number.  
*Conclusion:* There is a positive integer  $n$  such that  $n > x$ .  
*Converse:* If there is a positive integer  $n$  such that  $n > x$ , then  $x$  is a real number.  
*Contrapositive:* If for all positive integers  $n$  we have that  $n \leq x$ , then  $x$  is not a real number.  
(b) *Hypotheses:*  $x + 5$  is rational [and  $x$  is real].  
*Conclusion:*  $x$  is rational.  
*Converse:* If [the real number]  $x$  is rational, then  $x + 5$  is rational.  
*Contrapositive:* If  $x + 5$  is irrational, then  $x$  is irrational.  
(c) *Hypotheses:*  $x$  is a natural number.  
*Conclusion:*  $x \geq 1$ .  
*Converse:* If  $x$  is [a real number] such that  $x \geq 1$ , then  $x$  is a natural number.  
*Contrapositive:* If [ $x$  is a real number such that]  $x < 1$ , then  $x$  is not a natural number.  
(d) *Hypotheses:*  $n \in \mathbb{Z}$  and  $n < 0$ .  
*Conclusion:*  $n \leq -1$ .  
*Converse:* If  $n \leq -1$ , then  $n \in \mathbb{Z}$  and  $n < 0$ .  
*Contrapositive:* If  $n > -1$ , then either  $n \notin \mathbb{Z}$  or  $n \geq 0$ .
4. [Since this is a “there is” statement, it suffices to exhibit the element that satisfies the required property.] We have that  $0 \in \mathbb{R}$ , and since  $\epsilon < 0$ , we can simply take  $a = 0$ .
5. [Again, since this is a “there is” statement, it suffices to exhibit the element that satisfies the required property.] Since  $\epsilon, 1/2 \in \mathbb{R}$ , and  $\mathbb{R}$  is closed under product, we have that  $\epsilon/2 \in \mathbb{R}$ . Moreover, since  $\epsilon > 0$ , we have that  $\epsilon/2$  is also positive. Finally, we have that  $\epsilon/2 < \epsilon$ . So, we can take  $a = \epsilon/2$ .
7. (a) True.  $n = 1$  works.

- (b) False. Since  $\mathbb{R}$  is closed under subtraction, given any  $x$ , there is a  $y$  such that  $x \geq y$  [note that this statement is the negation of the original statement], for instance, we can take  $y = x$ .
- (c) True. If  $x$  is positive [and hence  $x \in \mathbb{N}$ , as we are assuming that  $x \in \mathbb{Z}$ ], we can take  $y = x + 1$  [as  $1 \in \mathbb{N}$  and  $\mathbb{N}$  is closed under addition]. If  $x$  is negative, we can take  $y = 1$  [which is in  $\mathbb{N}$ ].
- (d) True. Just take  $x = -1$ . Since  $y$  is positive [by hypothesis], clearly  $x < y$ .
- (e) True. Given a real number  $y$ , take  $x = [y] - 1$ , where  $[y]$  is the greatest integer less than or equal to  $x$ . [This is a function that you have likely encountered in calculus. It is also sometimes called the *floor* of  $y$  and can also be denoted by  $\lfloor y \rfloor$ . For instance, we have  $[2] = [2.00001] = [2.5] = [2.99999] = 2$  and  $[-2] = [-1.99999] = [-1.5] = [-1.00001] = -2$ . So, if  $y$  is positive we just drop the decimal places. If  $y$  is negative and not an integer, we drop the decimal places and subtract one. If  $y$  is negative and integer, we just get  $y$  itself.]
- (f) True. Given  $y$ , take  $x$  equal to  $[y]$ . [Hence,  $x$  is an integer.] So,  $x \leq y$ . Now, for all  $\epsilon > 0$ , we have that  $x \leq y < y + \epsilon$ , and hence  $x < y + \epsilon$ .

## Homework 2

### Pg. 12

- 8.a)  $-A = [-3, 2)$ ,  $-C = (-3, 2]$ ,  $A \setminus B = \emptyset$ ,  $B \setminus A = (-e, -2] \cup (3, \pi)$ ,  $A^c = (-\infty, -2] \cup (3, \infty)$ ,  $(B \setminus C)^c = ((-e, -2) \cup [3, \pi))^c = (-\infty, -e] \cup [-2, 3) \cup [\pi, \infty)$ .
- 9.a) Let  $A = \{x \in \mathbb{R} : 2 < 5x + 2 \leq 7\}$ . We prove the two containments.

$[A \subseteq (0, 1]]$  Let  $x \in A$ . Then, by definition of  $A$  we have that  $5x + 2 > 2$ . By subtracting 2 from both sides, we obtain,  $5x > 0$ . Since  $5 > 0$ , we can divide the inequality by 5 to obtain  $x > 0$ .

Also, by definition of  $A$ , we must also have  $5x + 2 \leq 7$ . As above, we can subtract 2 and divide by 5, obtaining  $x \leq 1$ . Thus,  $x > 0$  and  $x \leq 1$ . Therefore,  $x \in (0, 1]$  by definition [of intervals].

$[(0, 1] \subseteq A]$  Let  $x \in (0, 1]$ . Then, by definition [of intervals] we have that  $x > 0$ . Since  $5 > 0$  we can multiply by 5 and divide by 2, obtaining then  $5x + 2 > 2$ .

But, we must also have [by definition of intervals] that  $x \leq 1$ . Again, multiplying by 5 and adding 2, we obtain  $5x + 2 \leq 7$ . Hence, we have  $2 < 5x + 2 \leq 7$ , and so  $x \in A$ .

- 10.** The easiest way to do this is to use the proposition that I proved in class which states that “ $A$  is symmetric if, and only if, for all  $x \in A$ , we have that  $-x \in A$ ”. So, to prove that a set is symmetric, suffices to prove that for every  $x$  in the set,  $-x$  is also in the set.

Let  $x \in A \setminus B$ . Then, by definition,  $x \in A$  and  $x \notin B$ . Since  $A$  is symmetric and  $x \in A$ , we must have that  $-x \in A$ . Now, since  $B$  is also symmetric, if  $-x \in B$ , then  $-(-x) = x \in B$ . But, we have that  $x \notin B$ , and hence we cannot have that  $-x \in B$ . Therefore, we have that  $-x \in A \setminus B$  [as we just shown that  $-x \in A$  and  $-x \notin B$ ]. Thus,  $A \setminus B$  is symmetric.

- 12.** Let  $a$  and  $b$  be even integers. Then, by definition, there are integers  $m$  and  $n$  such that  $a = 2m$  and  $b = 2n$ . Then,  $ab = (2m)(2n) = 2(2mn)$ . Since  $2, m, n \in \mathbb{Z}$  and  $\mathbb{Z}$  is closed under products, we have that  $2mn \in \mathbb{Z}$ . Thus, by definition of even, we have that  $ab$  is even.

- 15.** Let  $a$  be an odd integer and  $b$  be an even integer. Then, by definition, there are  $m, n \in \mathbb{Z}$ , such that  $a = 2m + 1$  and  $b = 2n$ . Then,  $a + b = (2m + 1) + 2n = 2(m + n) + 1$ . Since  $m, n \in \mathbb{Z}$  and  $\mathbb{Z}$  is closed under addition, we have that  $m + n \in \mathbb{Z}$ . Thus, by definition of odd, we have that  $a + b$  is odd.

- 16.** Let  $a$  be an odd integer and  $b$  be an even integer. Then, by definition, there are  $m, n \in \mathbb{Z}$ , such that  $a = 2m + 1$  and  $b = 2n$ . Then,  $ab = (2m + 1)(2n) = 2(mn + n)$ . Since  $m, n \in \mathbb{Z}$  and  $\mathbb{Z}$  is closed under addition and product, we have that  $mn + n \in \mathbb{Z}$ . Thus, by definition of even, we have that  $ab$  is even.

- 17.** As in problem 10, we use the proposition from class.

Let  $E$  be the set of all even integers and  $a \in E$ . Then, by definition,  $a = 2m$  for some  $m \in \mathbb{Z}$ . Thus,  $-a = -(2m) = 2(-m)$ . Now, since  $0, m \in \mathbb{Z}$  and  $\mathbb{Z}$  is closed under subtractions, we have that  $-m \in \mathbb{Z}$ . Therefore, we have, by definition of even, that  $-a \in E$ . Hence,  $E$  is symmetric.

# Homework 3

Pg. 17

24.(c) (i)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ :

[“ $\subseteq$ ”:] Let  $x \in A \cup (B \cap C)$ . Then either  $x \in A$  or  $x \in B \cap C$  [and maybe in both at the same time]. [We have then two cases:  $x \in A$  and  $x \in B \cap C$ .] If  $x \in A$ , then by definition of union, we have that  $x \in A \cup B$  and  $x \in A \cup C$ , and therefore  $x \in (A \cup B) \cap (A \cup C)$  by the definition of intersection. [Which concludes the first case.] Now, if  $x \in B \cap C$ , then  $x \in B$  and  $x \in C$ . The former gives us that  $x \in A \cup B$ , and the latter gives us that  $x \in A \cup C$ . Therefore  $x \in (A \cup B) \cap (A \cup C)$ .

[“ $\supseteq$ ”:] Let  $x \in (A \cup B) \cap (A \cup C)$ . Hence,  $x \in (A \cup B)$  and  $x \in (A \cup C)$ . If  $x \in A$ , then clearly  $x \in A \cup (B \cap C)$ . If not, since  $x \in (A \cup B)$  and  $x \in (A \cup C)$ , we must have that  $x \in B$  [for the former to hold] and  $x \in C$  [for the latter to hold]. Therefore,  $x \in B \cap C$ , and so  $x \in A \cup (B \cap C)$ .

(i)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ :

[“ $\subseteq$ ”:] Let  $x \in A \cap (B \cup C)$ . Then  $x \in A$  and  $x \in B \cup C$ . So,  $x \in A$  and  $x$  is also in either  $B$  or  $C$ . If  $x \in B$ , then  $x \in A \cap B$ , and if  $x \in C$ , then  $x \in A \cap C$ . Therefore,  $x \in (A \cap B) \cup (A \cap C)$ .

[“ $\supseteq$ ”:] Let  $x \in (A \cap B) \cup (A \cap C)$ . Hence,  $x \in (A \cap B)$  or  $x \in (A \cap C)$ . [So, we have two cases.] If  $x \in A \cap B$ , then  $x \in A$  and  $x \in B$ . In particular,  $x \in B \cup C$ , and hence  $x \in A \cap (B \cup C)$ . On the other hand, if  $x \in A \cap C$ , then  $x \in A$  and  $x \in C$ . In particular,  $x \in B \cup C$ , and hence  $x \in A \cap (B \cup C)$ . Hence, in either case, we have that  $x \in (A \cap B) \cup (A \cap C)$ .

25.(b) We use the form of De Morgan’s Law that we have already proved, namely:

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \quad \text{and} \quad A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

(i)  $(A \cup B)^c = A^c \cap B^c$ :

We have

$$\begin{aligned} (A \cup B)^c &= \mathbb{R} \setminus (A \cup B) && \text{[by defn. of complement]} \\ &= (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B) && \text{[by De Morgan’s Law (proved)]} \\ &= A^c \cap B^c. && \text{[by defn. of complement]} \end{aligned}$$

(ii)  $(A \cap B)^c = A^c \cup B^c$ :

We have

$$\begin{aligned}(A \cap B)^c &= \mathbb{R} \setminus (A \cap B) && \text{[by defn. of complement]} \\ &= (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B) && \text{[by De Morgan's Law (proved)]} \\ &= A^c \cup B^c. && \text{[by defn. of complement]}\end{aligned}$$

26. [“ $\Rightarrow$ ”:] Assume that  $A \cap B = B$  and let  $x \in B$ . Then  $x \in A \cap B$ , and hence  $x \in A$ .

[“ $\Leftarrow$ ”:] Assume that  $B \subseteq A$ .

[“ $\subseteq$ ”:] Let  $x \in A \cap B$ . Then,  $x \in A$  and  $x \in B$ . In particular,  $x \in B$ .

[“ $\supseteq$ ”:] Let  $x \in B$ . Then, since  $B \subseteq A$ , we also have that  $x \in A$ .

Thus,  $x \in A$  and  $x \in B$ , i.e.,  $x \in A \cap B$ .

28. (a)  $x > 2$  or  $x < 1$ .

(b)  $x \leq 1$  and  $x \geq -1$  [or  $-1 \leq x \leq 1$ ].

(c)  $x \leq 5$  or  $x \leq 3$ .

## Pg. 20

29. The statement is true.

“ $\subseteq$ ”:] Let  $x \in A \cap B$ . Then,  $x \in A$  and  $x \in B$ . Therefore, since  $x \in A$ , we have that  $x \notin A^c$  [by definition of complement], and thus [since  $x \in B$ ]  $x \in B \setminus A^c$ . Since also  $x \in A$ , we have  $x \in A \cap (B \setminus A^c)$ .

[“ $\supseteq$ ”:] Let  $x \in A \cap (B \setminus A^c)$ . Then,  $x \in A$  and  $x \in B \setminus A^c$ . The latter means that  $x \in B$  and  $x \notin A^c$ . Therefore,  $x \in A$  and  $x \in B$ , and hence  $x \in A \cap B$ .  $\square$

31. The statement is true.

“ $\subseteq$ ”:] Let  $x \in (A \cap B) \cup (A \setminus B)$ . Then, either  $x \in A \cap B$  or  $x \in A \setminus B$ . The former means that  $x \in A$  and  $x \in B$ , the latter means that  $x \in A$  and  $x \notin B$ . But, in either case,  $x \in A$ .

[“ $\supseteq$ ”:] Let  $x \in A$ . We must have that either  $x \in B$  or  $x \notin B$ . If the former holds, we have that  $x \in A \cap B$ . If the latter holds, then  $x \in A \setminus B$ . Therefore,  $x \in (A \cap B) \cup (A \setminus B)$ .  $\square$

- 32.** The statement is false. Take  $A = B = \{1\}$ . Then,  $A \cap B = \{1\}$ . But,  $A \setminus B = B \setminus A = \emptyset$ . Hence,  $(A \setminus B) \cup (B \setminus A) = \emptyset \neq \{1\} = A \cup B$ .
- 33.** The statement is false, since for the book “ $\subset$ ” means “ $\subsetneq$ ”. For a counterexample, take  $A = B = \{1\}$ . Then,  $B \setminus A = \emptyset$ , but  $B \not\subsetneq A$ , as  $A = B$ . [If you replace  $\subsetneq$  by  $\subseteq$ , then the statement becomes true.]
- 36.** Suppose that  $A \cap B = A \cap C$  and that  $A \cup B = A \cup C$ .
- [“ $\subseteq$ ”:] Let  $x \in B$ . Then,  $x \in A \cup B = A \cup C$ . Then, either  $x \in A$  or  $x \in C$ . [We need to show that  $x \in C$ .] If the latter holds, then  $x \in C$ , and we are done. So, suppose that the former holds. Then,  $x \in B$  and  $x \in A$ . Therefore,  $x \in A \cap B = A \cap C$ . Thus,  $x \in C$ .
- [“ $\supseteq$ ”:] [This is completely symmetrical to the above!] Let  $x \in C$ . Then,  $x \in A \cup C = A \cup B$ . Then, either  $x \in A$  or  $x \in B$ . [We need to show that  $x \in B$ .] If the latter holds, then  $x \in B$ , and we are done. So, suppose that the former holds. Then,  $x \in C$  and  $x \in A$ . Therefore,  $x \in A \cap C = A \cap B$ . Thus,  $x \in B$ .
- 37.** Suppose that  $m/n$  is an odd integer. Then,  $m/n = 2k + 1$  for some  $k \in \mathbb{Z}$ . Therefore,  $m = n(2k + 1)$ . Then, if  $n$  is odd, then  $m$  is also odd as a product of odds is odd. [See below.] If  $n$  is even, then  $m$  is even as product of even and odd is even [Problem 16 on pg. 13, done above].

*Product of odds is odd:* Let  $a$  and  $b$  be odds. Then, there are  $m, n \in \mathbb{Z}$  such that  $a = 2m + 1$  and  $b = 2n + 1$ . Then,  $ab = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$ . Since  $m, n \in \mathbb{Z}$ , we have that  $(2mn + m + n) \in \mathbb{Z}$ , and hence  $ab$  is odd.

## Homework 4

### Pg. 24

2. (a) The statements are true. [To prove that a set is empty, we need to show that there is no element in it. Usually, you do it by contradiction, assuming that there is an element in it.]

Suppose  $(x, y) \in \emptyset \times B$ . Then, by definition of Cartesian product,  $x \in \emptyset$  and  $y \in B$ . But we cannot have  $x \in \emptyset$ , as it is empty. Thus, we cannot have  $(x, y) \in \emptyset \times B$ , i.e.,  $\emptyset \times B = \emptyset$ .

[The proof of  $A \times \emptyset = \emptyset$  is similar.]

(b) True.

[“ $\subseteq$ ”:] Let  $(x, y) \in (A \cup B) \times C$ . Then, by definition of Cartesian product, we have that  $x \in A \cup B$  and  $y \in C$ . So,  $(x \in A \text{ or } x \in B)$  and  $y \in C$ . Suppose that  $x \in A$ . Then, since  $y \in C$ , we have that  $(x, y) \in A \times C$ . If, on the other hand,  $x \in B$ , then  $(x, y) \in B \times C$ . So,  $(x, y) \in A \times C$  or  $(x, y) \in B \times C$ . Hence,  $(x, y) \in (A \times C) \cup (B \times C)$ .

[“ $\supseteq$ ”:] Let  $(x, y) \in (A \times C) \cup (B \times C)$ . Then,  $(x, y) \in A \times C$  or  $(x, y) \in B \times C$ . If the former holds, then  $x \in A$  and  $y \in C$ . Thus,  $x \in A \cup B$  [by definition of union] and  $y \in C$ , and hence  $(x, y) \in (A \cup B) \times C$  [by definition of the Cartesian product].

Now, if  $(x, y) \in B \times C$ , then  $x \in B$  and  $y \in C$ . Thus,  $x \in A \cup B$  [by definition of union] and  $y \in C$ , and hence  $(x, y) \in (A \cup B) \times C$  [by definition of the Cartesian product].

(c) True.

[“ $\subseteq$ ”:] Let  $(x, y) \in (A \cap B) \times (C \cap D)$ . Then, by definition of Cartesian product, we have that  $x \in A \cap B$  and  $y \in C \cap D$ . Thus, by definition of intersection, we have that  $(x \in A \text{ and } x \in B)$  and  $(y \in C \text{ and } y \in D)$ . [Note that, since we have only and’s and no or’s, we do not need the parenthesis.] Hence,  $(x \in A \text{ and } y \in C)$  and  $(x \in B \text{ and } y \in D)$ . So, by definition of intersection, we have that  $(x, y) \in (A \times C) \cap (B \times D)$ .

[“ $\supseteq$ ”:] Let  $(x, y) \in (A \times C) \cap (B \times D)$ . Then, by definition intersection, we have that  $(x, y) \in A \times C$ , and  $(x, y) \in B \times D$ . Thus, by definition of Cartesian product, we have that  $(x \in A \text{ and } y \in C)$  and  $(x \in B \text{ and } y \in D)$ . [Note that, since we have only and’s and no or’s, we do not need the parenthesis.] Hence,  $(x \in A \text{ and } x \in B)$  and  $(y \in C \text{ and } y \in D)$ . So, by definition of intersection,  $x \in A \cap B$  and  $y \in C \cap D$ . Finally, by the definition of Cartesian product we have that  $(x, y) \in (A \cap B) \times (C \cap D)$ .

- (d) False. Let  $A = C = \{1\}$ ,  $B = D = \{2\}$ . Then,  $A \cup B = C \cup D = \{1, 2\}$ . Thus,  $(A \cup B) \times (C \cup D) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ . On the other hand,  $A \times C = B \times D = \{(1, 2)\}$ , and so  $(A \times C) \cup (B \times D) = \{(1, 2)\}$ . Therefore, the two sets are not equal in this case.
3. (a) Solving for  $y$  we get  $y = \pm\sqrt{x^2 - 1}$ . So, we must have  $x^2 - 1 \geq 0$ , and hence  $x \in (-\infty, -1] \cup [1, \infty)$ . So, the domain is  $(-\infty, -1] \cup [1, \infty)$ .  
Now, solving for  $x$ , we obtain  $x = \pm\sqrt{1 + y^2}$ . Thus, we can take any  $y$ , i.e., the range is  $\mathbb{R}$ .
- (b) Solving for  $y$ , we get  $y = (x^2 - 1)/x$ . So, if  $x \neq 0$ , we can obtain a  $y$ . If  $x = 0$ , we would have  $0 = -1$ , and hence  $x = 0$  is not in the domain. So, the domain is  $\mathbb{R} \setminus \{0\}$ .  
We have that  $x^2 - yx - 1 = 0$ . Hence, we can use the quadratic formula to solve for  $x$ , obtaining  $x = (y \pm \sqrt{y^2 + 4})/2$ . Hence, for all  $y$ , we can find an  $x$ . Therefore, the range is  $\mathbb{R}$ .

## Pg. 28

4. (a) [Reflexive:] Since  $x - x = 0$  is nonnegative, we have that  $x\mathcal{R}x$ , and hence the relation is reflexive.  
[Symmetric:] The relation is not symmetric, as  $2\mathcal{R}1$ , as  $2 - 1 = 1 \geq 0$ , but  $1 - 2 = -1 < 0$ , and hence  $1\not\mathcal{R}2$  does not hold.  
[Transitive:] Suppose that  $x\mathcal{R}y$  and  $y\mathcal{R}z$ . Then  $x - y \geq 0$  and  $y - z \geq 0$ . So,  $(x - y) + (y - z) = x - z \geq 0$  [as sum of nonnegatives is nonnegative]. Thus,  $x\mathcal{R}z$ , and the relation is transitive.
- (b) Since  $\mathbb{N}$  is closed under multiplication and  $\mathbb{N} \subseteq \mathbb{Z}$ , the product of any  $x, y \in \mathbb{N}$  is in  $\mathbb{Z}$ . Thus, every element is related to each other. Therefore, clearly it is an equivalence relation.
- (c) [Reflexive:] The relation is not reflexive, as  $x - x = 0$ , which is even, and hence  $x\mathcal{R}x$  never holds.  
[Symmetric:] Suppose that  $x\mathcal{R}y$ . Then  $x - y$  is odd. So,  $x - y = 2k + 1$ , for some  $k \in \mathbb{Z}$ . Thus,  $y - x = -2k - 1 = 2(-k - 1) + 1$ . Since  $k \in \mathbb{Z}$ , we have that  $(-k - 1) \in \mathbb{Z}$ , and thus  $y - x$  is odd. Therefore,  $y\mathcal{R}x$ , and the relation is symmetric.

[Transitive:] The relation is not transitive. For example,  $4\mathcal{R}3$ , as  $4 - 3 = 1$  is odd, and  $3\mathcal{R}2$ , as  $3 - 2 = 1$  is also odd. But  $4 - 2$  is even, and hence  $4\mathcal{R}2$  does not hold.

- (d) [Reflexive:] The relation is not reflexive, as  $x - x = 0$ , which is rational, and hence  $x\mathcal{R}x$  never holds.

[Symmetric:] Suppose that  $x\mathcal{R}y$ . Then  $x - y$  is irrational. Suppose now that  $y - x$  is rational. [We need to derive a contradiction.] Then  $y - x = m/n$ , with  $m, n \in \mathbb{Z}$  and  $n \neq 0$ . But then,  $x - y = -(y - x) = (-m)/n$ , where,  $-m, n \in \mathbb{Z}$  and  $n \neq 0$ . Therefore, we would have that  $x - y$  is also rational, which contradicts the assumption that it was irrational. Therefore,  $y - x$  must be irrational and hence  $y\mathcal{R}x$ , and the relation is symmetric.

[Transitive:] The relation is not transitive. For example,  $\sqrt{2} + 1\mathcal{R}1$ , as  $(\sqrt{2} + 1) - 1 = \sqrt{2}$  is irrational [assuming this to be true, as we've discussed in class early in the semester], and  $1\mathcal{R}(\sqrt{2} + 1)$ , as  $1 - (\sqrt{2} + 1) = -\sqrt{2}$  is irrational. But  $(\sqrt{2} + 1) - (\sqrt{2} - 1) = 0$  is rational, and hence  $(\sqrt{2} + 1)\mathcal{R}(\sqrt{2} + 1)$  does not hold. [Don't get confused here. We were not proving that it is not symmetric here!]

5. (d) [Reflexive:] Given  $x \in \mathbb{R}$ , we have that  $x - x = 0 \in \mathbb{Z}$ . Thus,  $x \sim x$  for all  $x \in \mathbb{R}$ .

[Symmetric:] Assume that  $x \sim y$ . [We need to show that  $y \sim x$ .] Then, by definition, we have that  $x - y \in \mathbb{Z}$ . But then, [as  $0 \in \mathbb{Z}$  and  $\mathbb{Z}$  is closed under differences] we have that  $-(x - y) = y - x \in \mathbb{Z}$ , and thus  $y \sim x$  [by definition].

[Transitive:] Suppose that  $x \sim y$  and  $y \sim z$ . [We need to show that  $x \sim z$ .] Then,  $x - y, y - z \in \mathbb{Z}$ . Since  $\mathbb{Z}$  is closed under addition, we have that  $(x - y) + (y - z) = x - z \in \mathbb{Z}$ . Therefore,  $x \sim z$ .

Now, given  $x \in \mathbb{R}$ , we have that

$$\begin{aligned}\bar{x} &= \{y \in \mathbb{R} : x \sim y\} \\ &= \{y \in \mathbb{R} : x - y = k \text{ for some } k \in \mathbb{Z}\} \\ &= \{x + k : k \in \mathbb{Z}\}\end{aligned}$$

Hence, for example, we have  $\bar{0} = \mathbb{Z}$ ,  $\overline{0.5} = \{\dots, -1.5, -0.5, 0.5, 1.5, 2.5, \dots\}$ ,  $\overline{0.1} = \{\dots, -1.9, -0.9, 0.1, 1.1, 2.1, \dots\}$ .

[Also, observe that given  $x \in \mathbb{R}$ , let  $y \stackrel{\text{def}}{=} x - [x]$ . Then,  $y$  is the decimal part of  $x$  and  $y \in [0, 1)$ . Then,  $\bar{x} = \bar{y}$ , as  $x - y = [x] \in \mathbb{Z}$ , and so  $x \sim y$ . Also observe

that if  $x, y \in [0, 1)$ , with  $x \neq y$ , then  $\bar{x} \neq \bar{y}$ , as  $0 < |x - y| < 1$ , and hence  $x \approx y$ . Therefore, for each real number  $x$ , we have that there exists a *unique* element  $y$  of  $[0, 1)$  such that  $x \in \bar{y}$ .]

(e) [Reflexive:] Given  $x \in \mathbb{Z}$ , we have that  $x = x + 3 \cdot 0$ , and  $0 \in \mathbb{Z}$ . Thus,  $x \sim x$  for all  $x \in \mathbb{Z}$ .

[Symmetric:] Assume that  $x \sim y$ . [We need to show that  $y \sim x$ .] Then, by definition, we have that  $x = y - 3k$ , for some  $k \in \mathbb{Z}$ . But then, solving for  $y$  we have that  $y = x + 3k = x - 3 \cdot (-k)$ , and  $-k \in \mathbb{Z}$  [as  $0 \in \mathbb{Z}$  and  $\mathbb{Z}$  is closed under differences]. Thus  $y \sim x$  [by definition].

[Transitive:] Suppose that  $x \sim y$  and  $y \sim z$ . [We need to show that  $x \sim z$ .] Then,  $x = y - 3k$  and  $y = z - 3l$ , for some  $k, l \in \mathbb{Z}$ . [Careful to not repeat letters here!] Replacing the formula for  $y$  of the second equation in the first, we obtain  $x = z - 3l - 3k = z - 3(k + l)$ . Since  $\mathbb{Z}$  is closed under addition, we have that  $k + l \in \mathbb{Z}$ . Therefore,  $x \sim z$ .

Now, given  $x \in \mathbb{Z}$ , we have that

$$\begin{aligned} \bar{x} &= \{y \in \mathbb{R} : x \sim y\} \\ &= \{y \in \mathbb{R} : x = y - 3k \text{ for some } k \in \mathbb{Z}\} \\ &= \{y \in \mathbb{R} : y = x + 3k \text{ for some } k \in \mathbb{Z}\} \\ &= \{x + 3k : k \in \mathbb{Z}\} \end{aligned}$$

Thus, we have:

$$\begin{aligned} \bar{0} &= \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}, \\ \bar{1} &= \{\dots, -8, -5, -2, 1, 4, 7, 10, \dots\}, \\ \bar{2} &= \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}. \end{aligned}$$

Hence, these are all equivalence classes [as each integer is in exactly one of those three sets.]

[Note that the first set is the set of all integers of the form  $3k$ , i.e., all multiples of 3, i.e., all integers divisible by 3. The second is the set of all integers of the form  $3k + 1$ , i.e., all integers that have remainder 1 when divided by 3. The third set is the set of all integers of the form  $3k + 2$ , i.e., all integers that have remainder 2 when divided by 3.]

6. Since  $X = \{x\}$ , we have that  $X \times X = \{(x, x)\}$ . Therefore, the only nonempty subset of  $X \times X$  is  $X \times X$  itself, and hence the only nonempty relation on  $X$  is  $\mathcal{R} = \{(x, x)\} = X \times X$ . Then, by definition,  $x\mathcal{R}x$ .

[Reflexive:] We have to show that for all  $y \in X$ , we have  $y\mathcal{R}y$ . But, there is only  $x \in X$ , and we do have  $x\mathcal{R}x$ , as noted above. Hence, the relation is reflexive.

[Symmetric:] Suppose  $y\mathcal{R}z$ . [We need to show that  $z\mathcal{R}y$ .] But, the only possibility for  $y$  and  $z$  is  $x$ , and hence  $z\mathcal{R}y$  is the same as  $x\mathcal{R}x$ , which is true [as observed above]. Hence, the relation is symmetric.

[Transitive:] Suppose  $y\mathcal{R}z$  and  $z\mathcal{R}w$ . [We need to show that  $y\mathcal{R}w$ .] But, the only possibility for  $y$ ,  $z$ , and  $w$  is  $x$ , and hence  $y\mathcal{R}w$  is the same as  $x\mathcal{R}x$ , which is true [as observed above]. Hence, the relation is transitive.

## Homework 5

### Pg. 28

7. [I proved that in class. Here it is again.] Suppose that  $x \sim y$ .

[“ $\subseteq$ ”:] Let  $z \in \bar{x}$ . Then, by definition,  $z \sim x$ . Now, since the relation is transitive and we have  $x \sim y$ , we have that  $z \sim y$ . Therefore  $z \in \bar{y}$ .

[“ $\supseteq$ ”:] Let  $z \in \bar{y}$ . Then, by definition,  $z \sim y$ . Now, since the relation is symmetric and we have  $x \sim y$ , we also have  $y \sim x$ . Since the relation is transitive and we have  $z \sim y$  and  $y \sim x$ , we have that  $z \sim x$ . Therefore  $z \in \bar{x}$ .

9.  $\mathcal{R} \cap \mathcal{S}$  is an equivalence relation. Let  $\mathcal{T}$  denote  $\mathcal{R} \cap \mathcal{S}$ . [Remember that  $x\mathcal{R}y$  simply means that  $(x, y) \in \mathcal{R}$ . We shall use this interpretation below.]

[Reflexive:] Since  $\mathcal{R}$  and  $\mathcal{S}$  are reflexive, for all  $x \in X$ , we have that  $(x, x) \in \mathcal{R}$  and  $(x, x) \in \mathcal{S}$ . Therefore,  $(x, x) \in \mathcal{R} \cap \mathcal{S} = \mathcal{T}$ .

[Symmetric:] Suppose that  $(x, y) \in \mathcal{T}$ . [We need to show that  $(y, x) \in \mathcal{T}$ .] Then,  $(x, y) \in \mathcal{R}$  and  $(x, y) \in \mathcal{S}$  [since  $\mathcal{R} \cap \mathcal{S} = \mathcal{T}$ ]. Since  $\mathcal{R}$  and  $\mathcal{S}$  are symmetric, we have that  $(y, x) \in \mathcal{R}$  and  $(y, x) \in \mathcal{S}$ . Therefore,  $(y, x) \in \mathcal{R} \cap \mathcal{S} = \mathcal{T}$ .

[Transitive:] Suppose that  $(x, y), (y, z) \in \mathcal{T}$ . [We need to show that  $(x, z) \in \mathcal{T}$ .] Then,  $(x, y), (y, z) \in \mathcal{R}$  and  $(x, y), (y, z) \in \mathcal{S}$  [since  $\mathcal{R} \cap \mathcal{S} = \mathcal{T}$ ]. Since  $\mathcal{R}$  and  $\mathcal{S}$  are transitive,

we have that  $(x, z) \in \mathcal{R}$  and  $(x, z) \in \mathcal{S}$ . Therefore,  $(x, z) \in \mathcal{R} \cap \mathcal{S} = \mathcal{T}$ .

The union is not an equivalence relation in general. It is always reflexive and symmetric, but it may be nontransitive. For example let  $\mathcal{R}$  and  $\mathcal{S}$  be the relations on  $\mathbb{Z}$  defined by  $x\mathcal{R}y$  if  $x - y$  is even [as in Example 2.16 from the text] and  $x\mathcal{S}y$  if  $x = y - 3k$  for some integer  $k$  [as in Problem 5(e)]. Let  $\mathcal{T}$  be  $\mathcal{R} \cup \mathcal{S}$ . Then,  $(1, 3) \in \mathcal{R}$  [as  $1 - 3 = -2$  is even] and  $(3, 6) \in \mathcal{S}$  [as  $3 = 6 - 3 \cdot 1$ ]. Thus,  $(1, 3), (3, 6) \in \mathcal{T}$ . But  $1 - 6 = -5$  is not even and  $1 \neq 6 - 3k$ , for otherwise we would have that  $5 = 3k$  for some integer  $k$ , but 5 is not divisible by 3. Therefore,  $(1, 6) \notin \mathcal{T}$ , and hence  $\mathcal{T}$  is *not* transitive.

10. Since we are assuming  $\mathcal{R}$  is reflexive we just need to show that it is symmetric and transitive.

[Symmetric:] Suppose that  $x\mathcal{R}y$ . Since  $\mathcal{R}$  is symmetric, we have that  $x\mathcal{R}x$ . Hence, we have  $x\mathcal{R}y$  and  $x\mathcal{R}x$ , and by the property given in the statement [taking  $z = x$ ], gives us that  $y\mathcal{R}x$ .

[Transitive:] Assume that  $x\mathcal{R}y$  and  $y\mathcal{R}z$ . Since we just proved that  $\mathcal{R}$  is symmetric, we have also  $y\mathcal{R}x$ . But, the property given in the statement then tells us that since  $y\mathcal{R}x$  and  $y\mathcal{R}z$ , we must have  $x\mathcal{R}z$ .

Now assume that  $\mathcal{R}$  is an equivalence relation. We want to prove that if  $x\mathcal{R}y$  and  $x\mathcal{R}z$ , then  $y\mathcal{R}z$ . Indeed, assume  $x\mathcal{R}y$  and  $x\mathcal{R}z$ . Then, since  $\mathcal{R}$  is symmetric, we have that  $y\mathcal{R}x$ . Since it is transitive, we have that [since  $y\mathcal{R}x$  and  $x\mathcal{R}z$ ]  $y\mathcal{R}z$ .

[Note that we proved that  $\mathcal{R}$  is an equivalence relation if, and only if,

- $\mathcal{R}$  is reflexive [i.e., for all  $x \in X$ , we have  $x\mathcal{R}x$ ];
- if  $x\mathcal{R}y$  and  $x\mathcal{R}z$ , then  $y\mathcal{R}z$ .

Since this is an if-and-only-if, one could replace the original definition of equivalence relation by this new one, i.e., it suffices to check that these two properties above instead of the three from the original definition when checking if a relation is an equivalence relation.]

# Homework 6

## Pg. 33

1. In all of these, you *have* to draw the graph to be able to read the answer from it!  
[Alternatively, one could use calculus, but it is easier to just draw it.]

(a)  $f(A) = (-7, -1]$ ,  $f^{-1}(B) = [2, 8/3]$ .

(b)  $f(A) = [1, 2)$ ,  $f^{-1}(B) = \{0\}$ .

(c)  $f(A) = (-\infty, 0)$ ,  $f^{-1}(B) = (1, \infty)$ .

2. [Reflexive:] Let  $x \in X$ . Then, clearly  $f(x) = f(x)$  [as  $f$  is a function, there is a *single possible output* for each  $x \in X$ ]. Hence,  $x \sim x$ .

[Symmetric:] Suppose that  $x \sim y$ . Then  $f(x) = f(y)$  by definition. Thus, [since equality is symmetric] we have  $f(y) = f(x)$ , and hence  $y \sim x$  by definition.

[Transitive:] Suppose that  $x \sim y$  and  $y \sim z$ . Then, by definition, we have  $f(x) = f(y)$  and  $f(y) = f(z)$ . So, [as equality is transitive] we have that  $f(x) = f(z)$ , i.e.,  $x \sim z$ .

We have

$$\bar{x} = \{x' \in X : f(x') = f(x)\}.$$

[Hence, the equivalence class of an element is the set of all other elements in  $X$  which have the same output via  $f$ .]

3. (a) [**Note:** To prove that a set is empty, we assume that there is an element in the set and derive a contradiction.]

Let  $x \in f^{-1}(\emptyset)$ . Then, by definition, we have that  $f(x) \in \emptyset$ , which is impossible [as  $\emptyset$  has no elements!]. Therefore,  $f^{-1}(\emptyset) = \emptyset$ .

- (b) Let  $x \in f^{-1}(B)$ . Then, by definition, we have that  $f(x) \in B$ . Since  $B \subseteq A$ , we also have that  $f(x) \in A$ . Therefore, by definition of inverse image,  $x \in f^{-1}(A)$ .

- 4.(a) [“ $\subseteq$ ”:] Let  $x \in f^{-1}(A \cap B)$ . Then, by definition,  $f(x) \in A \cap B$ . So,  $f(x) \in A$  and  $f(x) \in B$ . By definition of inverse image, the former says that  $x \in f^{-1}(A)$  and the latter says that  $x \in f^{-1}(B)$ . Therefore,  $x \in f^{-1}(A) \cap f^{-1}(B)$ .

[“ $\supseteq$ ”:] Let  $x \in f^{-1}(A) \cap f^{-1}(B)$ . Then,  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ . So, by definition, we have that  $f(x) \in A$  and  $f(x) \in B$ , i.e.,  $f(x) \in A \cap B$ . So, by definition of inverse image again, we have that  $x \in f^{-1}(A \cap B)$ .

5. (a) Suppose that  $y \in f(\emptyset)$ . Then, by definition, there is  $x \in \emptyset$  such that  $y = f(x)$ . But this is impossible, as  $\emptyset$  has no elements. Thus,  $f(\emptyset) = \emptyset$ .
- (b) Let  $y \in f(B)$ . Then, by definition, there exists  $x \in B$  such that  $y = f(x)$ . But, since  $B \subseteq A$ , this same  $x$  is also in  $A$ . Therefore, by definition of direct image, we have that  $y = f(x) \in f(A)$ .
- 6.(b) Let  $y \in f(A) \setminus f(B)$ . So,  $y \in f(A)$  but  $y \notin f(B)$ . Thus, there exists  $x \in A$  such that  $y = f(x)$ , and for all  $x' \in B$ , we have that  $f(x') \neq y$ . So,  $x$  [the one in  $A$  above] is not in  $B$  as we do have  $y = f(x)$ . Therefore,  $x \in A \setminus B$ . Since  $y = f(x)$ , by definition of direct image, we have that  $f(x) \in f(A \setminus B)$ .

## Homework 7

### Pg. 38

8. In all of these you should graph the function. Then, you can use the *horizontal line test* to check if the function is one-to-one, and the techniques from the previous section to check onto [i.e.,  $f(A) = B$ ]. I will just post the results, but *you should always show work!*
- (a) One-to-one and onto, with inverse  $f^{-1}(x) = (x - 2)/3$ .
- (c) Not one-to-one [as, for instance,  $f(0) = f(2) = 0$ ], but onto. [On the quiz I mistyped  $A$  as  $[-1, 0]$  also, instead of  $[0, 2]$ . In that case, one can see that  $f$  is one-to-one, but not onto, as  $f([-1, 0]) = [0, 3] \neq [-1, 0]$ .]
- (e) Note that the graph of  $f(x)$  is the same as the line  $y = x + 1$ , but with a hole at  $x = 1$ . So,  $f(x)$  is not even defined in all of  $A$ , so it does not quite make sense to ask if  $f(x)$  is one-to-one on  $A$ . If we allow ourselves to disregard the whole, then the function would be one-to-one [on  $A \setminus \{1\}$ ]. In any case, it is certainly not onto  $B$  there is no  $x \in A$  such that  $f(x) = 2$ .
9. (a) [“ $\subseteq$ ”:] Follows from Theorem 3.11(b).
- [“ $\supseteq$ ”:] Let  $y \in f(A) \cap f(B)$ . Then,  $y \in f(A)$  and  $y \in f(B)$ . Hence, by definition of inverse image, there exists  $a \in A$  such that  $y = f(a)$  and there exists  $b \in B$  such that  $y = f(b)$ . [Careful to not use the same letter here for  $a$  and  $b$ !!! They

might be, in principle, different.] Thus, we have  $f(a) = f(b)$  [as both are equal to  $y$ ], and since  $f$  is one-to-one, we have  $a = b$ . So,  $a \in A$  and  $a \in B$ , i.e.,  $a \in A \cap B$ . Since  $y = f(a)$ , by definition of direct image, we have that  $y \in f(A \cap B)$ .

(b) [“ $\subseteq$ ”:] Let  $y \in f(A \setminus B)$ . Then, there exists  $x \in A \setminus B$  such that  $y = f(x)$ . So, since  $x \in A$ , we have that  $y \in f(A)$  [as  $y = f(x)$ ]. Also,  $y \notin f(B)$ , for if it were, we would have that there exists  $x' \in B$  such that  $y = f(x')$ . But, since  $y = f(x)$  and  $f$  is one-to-one, we have that  $x = x'$ . But this is a contradiction since  $x' \in B$ , while  $x = x' \notin B$ . Therefore, we cannot have  $y \in f(B)$ . Thus,  $y \in f(A) \setminus f(B)$ .

[“ $\supseteq$ ”:] Follows from Theorem 3.11(c).

(c) [“ $\subseteq$ ”:] Let  $x \in f^{-1}(f(A))$ . Then, by definition,  $f(x) \in f(A)$ . So, there is  $a \in A$  such that  $f(x) = f(a)$ . Since  $f$  is one-to-one, we have that  $x = a \in A$ .

[“ $\supseteq$ ”:] Follows from Theorem 3.15(b).

**10.** I accidentally proved these in class... But here it is anyway.

(a) Since  $f$  is a bijection [between  $A$  and  $f(A)$ ], it is invertible. Remember that if  $f$  is invertible, then so is  $f^{-1}$  [since  $f$  is an inverse of  $f^{-1}$  by definition]. Therefore, by Theorem 3.24,  $f^{-1}$  is a bijection [between  $f(A)$  and  $A$ ].

(b) This is just a repetition of (a). If  $f$  is onto  $B$ , then  $B = f(A)$  [by definition]. So, using (a), we have that  $f^{-1}$  is a bijection [between  $B = f(A)$  and  $A$ ].

## Homework 8

### Pg. 44

1. We prove it by induction on  $n$ . For  $n = 1$ , we have:

$$2 \cdot 1 - 1 = 1 = 1^2.$$

So, assume now that the statement holds for  $n$  [for some  $n \geq 1$ ], i.e.,

$$\sum_{k=1}^n (2k - 1) = n^2.$$

[We need to show that

$$\sum_{k=1}^{n+1} (2k - 1) = (n + 1)^2.]$$

We then have:

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \left[ \sum_{k=1}^n (2k-1) \right] + 2(n+1) - 1 \\ &= n^2 + 2n + 1 && \text{[by the IH]} \\ &= (n+1)^2.\end{aligned}$$

3. We prove it by induction on  $n$ . For  $n = 1$ , we have:

$$1^3 = (1 \cdot 2 \cdot 3)/6.$$

So, assume now that the statement holds for  $n$  [for some  $n \geq 1$ ], i.e.,

$$\sum_{k=1}^n k^3 = \left( \frac{n(n+1)}{2} \right)^2.$$

[We need to show that

$$\sum_{k=1}^{n+1} k^3 = \left( \frac{(n+1)(n+2)}{2} \right)^2.]$$

We then have:

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \left[ \sum_{k=1}^n k^3 \right] + (n+1)^3 \\ &= \left( \frac{n(n+1)}{2} \right)^2 + (n+1)^3 && \text{[by the IH]} \\ &= (n+1)^2 \cdot \frac{n^2 + 4(n+1)}{4} && \text{[factor } (n+1)^2\text{]} \\ &= \left( \frac{(n+1)(n+2)}{2} \right)^2.\end{aligned}$$

4. We prove it by induction on  $n$ . For  $n = 1$ , we have:

$$(2 \cdot 1 - 1)^2 = 1 = \frac{1 \cdot (4 \cdot 1^2 - 1)}{3}.$$

So, assume now that the statement holds for  $n$  [for some  $n \geq 1$ ], i.e.,

$$\sum_{k=1}^n (2k-1)^2 = \frac{n(4n^2-1)}{3}.$$

[We need to show that

$$\sum_{k=1}^{n+1} (2k-1)^2 = \frac{(n+1)(4(n+1)^2-1)}{3}.]$$

We then have:

$$\begin{aligned}
 \sum_{k=1}^{n+1} (2k-1)^2 &= \left[ \sum_{k=1}^n (2k-1)^2 \right] + (2(n+1)-1)^2 \\
 &= \frac{n(4n^2-1)}{3} + (2n+1)^2 && \text{[by the IH]} \\
 &= \frac{4n^3 - n + 3(4n^2 + 4n + 1)}{3} \\
 &= \frac{4n^3 + 12n^2 + 11n + 3}{3} \\
 &= \frac{(n+1)(4(n+1)^2 - 1)}{3}.
 \end{aligned}$$

[**Note:** The last equality is not immediate. The truth is that if we are careful with all our steps, we *should* have equality with what we needed. You can just expand the formulas in your scratch to check it.]

5. We prove it by induction on  $n$ . For  $n = 1$ , we have:

$$(a-1)\frac{1}{a} = 1 - \frac{1}{a}.$$

So, assume now that the statement holds for  $n$  [for some  $n \geq 1$ ], i.e.,

$$\sum_{k=1}^n (a-1)\frac{1}{a^k} = 1 - \frac{1}{a^n}.$$

[We need to show that

$$\sum_{k=1}^{n+1} (a-1)\frac{1}{a^k} = 1 - \frac{1}{a^{n+1}}.]$$

We then have:

$$\begin{aligned}
 \sum_{k=1}^{n+1} (a-1)\frac{1}{a^k} &= \left[ \sum_{k=1}^n (a-1)\frac{1}{a^k} \right] + (a-1)\frac{1}{a^{n+1}} \\
 &= 1 - \frac{1}{a^n} + \frac{1}{a^n} - \frac{1}{a^{n+1}} && \text{[by the IH]} \\
 &= 1 - \frac{1}{a^{n+1}}.
 \end{aligned}$$

## Homework 9

### Pg. 46

7. We prove it by induction on  $n$ . For  $n = 1$  we have  $1^5 - 1 = 0$  is divisible by 5. [**Note:** 0 is divisible by any non-zero integer, as  $0 = 0 \cdot n$  for all  $n$ .]

Now, assume that 5 divides  $n^5 - n$ . [We need to show that 5 also divides  $(n + 1)^5 - (n + 1)$ .] We then have,

$$\begin{aligned}(n + 1)^5 - (n + 1) &= (n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) - (n + 1) \\ &= 5 \cdot (n^4 + 2n^3 + 2n^2 + n).\end{aligned}$$

Since  $n^4 + 2n^3 + 2n^2 + n \in \mathbb{Z}$  [as  $n$  is  $\mathbb{Z}$ , and  $\mathbb{Z}$  is closed under products and addition], we have that 5 divides  $(n + 1)^5 - (n + 1)$ .

9. We prove it by induction on  $n$ . For  $n = 1$  we have  $7 + 13 = 20$  is divisible by 10.

Now, assume that 10 divides  $7^{2n-1} + 13^{2n-1}$ . [We need to show that 10 also divides  $7^{2(n+1)-1} + 13^{2(n+1)-1} = 7^{2n+1} + 13^{2n+1}$ .] We then have,

$$\begin{aligned}7^{2n+1} + 13^{2n+1} &= 7^2 \cdot 7^{2n-1} + 13^3 \cdot 13^{2n-1} \\ &= 7^2 \cdot (7^{2n-1} + 13^{2n-1}) + (13^3 - 7^2) \cdot 13^{2n-1} \\ &= 7^2 \cdot (7^{2n-1} + 13^{2n-1}) + 120 \cdot 13^{2n-1}.\end{aligned}$$

By the induction hypothesis, we have that 10 divides  $7^{2n-1} + 13^{2n-1}$ , and thus it also divides  $7^2 \cdot (7^{2n-1} + 13^{2n-1})$ . Since 10 divides 120, it also divides  $120 \cdot 13^{2n-1}$ . Thus, 10 divides  $7^{2n+1} + 13^{2n+1} = 7^2 \cdot (7^{2n-1} + 13^{2n-1}) + 120 \cdot 13^{2n-1}$  [since, in general, if  $n$  divides  $a$  and  $b$ , then  $n$  divides  $a + b$ ].

10. We prove it by induction on  $n$  that  $(n - 1)^3 + n^3 + (n + 1)^3 = 3n^3 + 6n$  is always divisible by 9 for  $n \geq 2$ . For  $n = 2$  we have  $1^3 + 2^3 + 3^3 = 36$  is divisible by 9.

Now assume that  $(n - 1)^3 + n^3 + (n + 1)^3 = 3n^3 + 6n$  is divisible by 9. [We need to show that  $n^3 + (n + 1)^3 + (n + 2)^3$  is also divisible by 9.] We have,

$$\begin{aligned}n^3 + (n + 1)^3 + (n + 2)^3 &= 3n^3 + 9n^2 + 15n + 9 \\ &= (3n^3 + 6n) + 9(n^2 + n).\end{aligned}$$

By the induction hypothesis, 9 divides  $(3n^3 + 6n)$ , and hence 9 divides  $n^3 + (n + 1)^3 + (n + 2)^3 = (3n^3 + 6n) + 9(n^2 + n)$  [since, in general, if  $n$  divides  $a$  and  $b$ , then  $n$  divides  $a + b$ ].

12. Note that we cannot use induction right away as it is not for all integers greater than some  $n_0$ , but for odds only. There are a couple of ways of doing this. One is to rephrase it as: “Prove that  $2^{2n-1} + 1$  is divisible by 3 for all positive integers  $n$ ”, as if

$n \in \{1, 2, 3, 4, 5, \dots\}$ , we have that  $2n - 1 \in \{1, 3, 5, 7, 9, \dots\}$ , i.e,  $2n - 1$  give us the positive odds when  $n$  run through the positive integers.

So, we prove this altered statement by induction on  $n$ . For  $n = 1$ , we have  $2^1 + 1 = 3$  is divisible by 3.

Now, assume that  $2^{2n-1} + 1$  is divisible by 3. [We need to show that  $2^{2(n+1)-1} + 1 = 2^{2n+1} + 1$  is also divisible by 3.] We have:

$$\begin{aligned} 2^{2n+1} + 1 &= 2^2 \cdot 2^{2n-1} + 1 \\ &= 2^2 \cdot 2^{2n-1} + 1 \\ &= 3 \cdot 2^{2n-1} + (2^{2n-1} + 1). \end{aligned}$$

Since, by induction hypothesis 3 divides  $2^{2n-1} + 1$  [and clearly 3 divides  $3 \cdot 2^{2n-1}$ ], we have that 3 divides  $2^{2n+1} + 1 = 3 \cdot 2^{2n-1} + (2^{2n-1} + 1)$  [since, in general, if  $n$  divides  $a$  and  $b$ , then  $n$  divides  $a + b$ ].

- 13.** We first note that  $k$  consecutive integers are of the form  $n + (n + 1) + \dots + (n + k - 1)$  [careful that it ends in  $(n + k - 1)$  not in  $(n + k)$ !] for some integer  $n$ .

We then prove that statement by induction on  $n$ . For  $n = 1$ , we have

$$1 + 2 + \dots + k = \frac{k(k + 1)}{2}$$

[as done in class, or on pg. 42 from the book]. Since  $k$  is odd,  $k + 1$  is even, and so  $(k + 1)/2$  is an integer. [Note that we don't always have that  $k$  divides  $k(k + 1)/2$ , for instance 2 does not divide  $2 \cdot 3/2 = 3$ . It is crucial that  $k$  is odd here, so that  $(k + 1)/2 \in \mathbb{Z}$ .] So, we have that  $k$  divides  $k \cdot ((k + 1)/2) = k(k + 1)/2$ .

Now assume that  $k$  divides  $n + (n + 1) + \dots + (n + k - 1)$ . [We must show that  $k$  also divides  $(n + 1) + (n + 2) + \dots + (n + k)$ .] We have

$$(n + 1) + (n + 2) + \dots + (n + k) = [n + (n + 1) + \dots + (n + k - 1)] + [(n + k) - n].$$

Since, by the induction hypothesis  $k$  divides the first bracket, and since the second bracket is equal to  $k$ , we have that  $k$  divides  $(n + 1) + (n + 2) + \dots + (n + k)$ .

[We could also have done the following:

$$\begin{aligned} \sum_{i=1}^k (n+k) &= \left[ \sum_{i=1}^k n \right] + \left[ \sum_{i=1}^k k \right] \\ &= n \left[ \sum_{i=1}^k 1 \right] + \frac{k(k+1)}{2} \\ &= n \cdot k + k \cdot \frac{k+1}{2}, \end{aligned}$$

which is divisible by  $k$  [using as above the fact that  $k$  is odd]. Note that we are using the the  $k$  consecutive integers are  $(n+1) + (n+2) + \dots + (n+k)$ , which is OK.]

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14. We prove the statements by induction on  $n$ . For the first, we have that  $2^0 = 1 \leq 1 = 1!$ .

Now, assume that  $2^{n-1} \leq n!$ . [We must prove that  $2^n \leq (n+1)!.$ ] We have

$$\begin{aligned} 2^n &= 2^{n-1} \cdot 2 \\ &\leq 2^{n-1} \cdot (n+1) && \text{[as } 2 \leq (n+1)\text{]} \\ &\leq n! \cdot (n+1) && \text{[by the IH]} \\ &= (n+1)! \end{aligned}$$

Hence,  $2^n \leq (n+1)!$ .

For the second statement, we have for  $n = 1$ , that  $2^1 + 1! = 3 < 6 = 3!$ .

Now, assume that  $2^n + n! \leq (n+2)!$ . [We must prove that  $2^{n+1} + (n+1)! \leq (n+3)!.]$

We have

$$\begin{aligned} 2^{n+1} + (n+1)! &= 2 \cdot 2^n + (n+1) \cdot n! \\ &\leq (n+1) \cdot [2^n \cdot n!] && \text{[as } 2 \leq (n+1)\text{]} \\ &\leq (n+1) \cdot (n+2)! && \text{[by the IH]} \\ &< (n+3) \cdot (n+2)! && \text{[as } (n+1) < (n+3)\text{]} \\ &= (n+3)! \end{aligned}$$

Hence,  $2^{n+1} + (n+1)! \leq (n+3)!$ .

**16.** We prove the inequality by induction on  $n$ . For  $n = 1$  we have  $1^{-2} = 1 \leq 1 = 2 - (1/1)$ .

Now assume that  $\sum_{k=1}^n k^{-2} \leq 2 - (1/n)$ . [We need to show that  $\sum_{k=1}^{n+1} k^{-2} \leq 2 - (1/(n+1))$ .] First observe that

$$\begin{aligned} \frac{1}{(n+1)^2} &\leq \frac{1}{n(n+1)} && \text{[as } n < (n+1) \text{ and we are dividing]} \\ &= \frac{1}{n} - \frac{1}{n+1}. \end{aligned}$$

Thus,

$$\frac{1}{(n+1)^2} - \frac{1}{n} \leq -\frac{1}{n+1}.$$

Now, with that, we have

$$\begin{aligned} \sum_{k=1}^{n+1} k^{-2} &= \left[ \sum_{k=1}^n k^{-2} \right] + \frac{1}{(n+1)^2} \\ &\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} && \text{[by the IH]} \\ &\leq 2 - \frac{1}{n+1} && \text{[by the above],} \end{aligned}$$

and hence  $\sum_{k=1}^{n+1} k^{-2} \leq 2 - (1/(n+1))$ .

[Of course, the proof above was “cleaned up”. We start with

$$\begin{aligned} \sum_{k=1}^{n+1} k^{-2} &= \left[ \sum_{k=1}^n k^{-2} \right] + \frac{1}{(n+1)^2} \\ &\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} && \text{[by the IH]} \end{aligned}$$

With that, we realize that we need to show that  $-1/n + 1/(n+1)^2 \leq 1/(n+1)$ , which is what we’ve done first.]

**17.** (a) We prove it by induction on  $n$ . For  $n = 1$ , we have  $(1+a)^1 \geq a1 + 1 \cdot a$ .

Now suppose that  $(1+a)^n \leq 1 + na$ . [We need to show that  $(1+a)^{n+1} \geq 1 + (n+1)a$ .] We have

$$\begin{aligned} (1+a)^{n+1} &= (1+a) \cdot (1+a)^n \\ &\geq (1+a) \cdot (1+na) && \text{[by the IH]} \\ &= 1 + na + a + na^2 \\ &\geq 1 + (n+1)a && \text{[as } na^2 > 0, \text{ since } a > 0]. \end{aligned}$$

Hence,  $(1+a)^{n+1} \geq 1 + (n+1)a$ .

(b) We just apply the above with  $a = 1/n$ . Since  $n$  is a positive integer, clearly  $1/n > 0$ , and we can indeed apply part (a). [It's very important to check that!!] Then, we have  $(1 + 1/n)^n \leq 1 + n \cdot (1/n) = 2$ .

[This is similar to the example done in class in which is easier to prove a more general statement than it is to prove a particular case of it. It would have been much harder to prove (b) directly!]

**19.** We will need the following: for all positive integers  $n$ , we have  $2n + 1 \leq 2^n$ . We prove this by induction on  $n$ . For  $n = 1$  we have  $2 \cdot 1 + 1 = 3 \leq 4 = 2^2$ .

Now assume that  $2n + 1 \leq 2^n$ . [We need to prove that  $2n + 3 \leq 2^{n+1}$ .] We have:

$$\begin{aligned} 2n + 3 &= (2n + 1) + 2 \\ &\leq 2^n + 2 && \text{[by the IH]} \\ &\leq 2^n + 2^n && \text{[as } 2, n \geq 1, 2 \leq 2^n\text{]} \\ &= 2^{n+1}. \end{aligned}$$

We now solve the actual problem. We prove it by induction on  $n$ . For  $n = 1$ , we have  $1^2 = 1 \leq 3 = 2^1 + 1$ .

Now assume that  $n^2 \leq 2^n + 1$ . We have:

$$\begin{aligned} (n + 1)^2 &= n^2 + 2n + 1 \\ &\leq (2^n + 1) + 2n + 1 && \text{[by the IH]} \\ &= (2^n + 1) + 2^n && \text{[by the above]} \\ &= 2^{n+1} + 1. \end{aligned}$$

[Again, this proof has been “cleaned up”. We first got

$$\begin{aligned} (n + 1)^2 &= n^2 + 2n + 1 \\ &\leq (2^n + 1) + 2n + 1 && \text{[by the IH].} \end{aligned}$$

Then, this tells us that to finish the proof we need to show that  $2n + 1 \leq 2^n$ , which is how we've started.]

# Homework 10

## Pg. 53

39. We have

$$\begin{aligned}x_0 &= 3 \\x_1 &= 3 + 1 \\x_2 &= 3 + 1 + 2 \\x_3 &= 3 + 1 + 2 + 3 \\&\vdots \\x_n &= 3 + 1 + 2 + \cdots + n = 3 + \frac{n(n+1)}{2}.\end{aligned}$$

Now, we must prove it by induction. For  $n = 0$  the formula clearly holds. Now, if  $x_n = 3 + n(n+1)/2$  for some  $n \geq 0$ , then by definition,

$$x_{n+1} = x_n + n = 3 + \frac{n(n+1)}{2} + (n+1) = 3 + \frac{(n+1)(n+2)}{2},$$

and hence the formula is valid for all  $n \geq 0$ .

40. We have

$$\begin{aligned}x_0 &= 1 \\x_1 &= 2 \cdot 1 + 1 = 2 + 1 \\x_2 &= 2 \cdot (2 + 1) + 1 = 2^2 + 2 + 1 \\x_3 &= 2 \cdot (2^2 + 2 + 1) = 2^3 + 2^2 + 2 + 1 \\&\vdots \\x_n &= 2 \cdot (2^{n-1} + \cdots + 2 + 1) = 2^n + \cdots + 2^2 + 2 + 1 \\&= \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1.\end{aligned}$$

[Here we used the fact that if  $a \neq 1$ , then

$$1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}.$$

This is well-known and often used in calculus or even pre-calculus. I mentioned but did not prove it in class. One can easily prove it by induction.]

Now, we must prove it by induction. For  $n = 0$  the formula clearly holds as  $1 = 2^1 - 1$ .

Now, if  $x_n = 2^{n+1} - 1$  for some  $n \geq 0$ , then by definition,

$$x_{n+1} = 2x_n + 1 = 2(2^{n+1} - 1) + 1 = 2^{n+2} - 1,$$

and hence the formula is valid for all  $n \geq 0$ .

**41.** We have

$$x_0 = 1$$

$$x_1 = 1$$

$$x_2 = 1 + 1 = 2$$

$$x_3 = 1 + 1 + 2 = 4$$

$$x_4 = 1 + 1 + 2 + 4 = 8$$

$$x_5 = 1 + 1 + 2 + 4 + 8 = 16$$

$\vdots$

So, one can guess that  $x_n = 2^{n-1}$  for  $n \geq 1$ , but not for  $n \geq 0$ . [Another way we can see it is

$$x_n = 1 + 1 + 2 + 2^2 + \cdots + 2^{n-2} = 1 + 2^{n-1} - 1 = 2^{n-1}]$$

Now, we must prove it by induction. For  $n = 1$  the formula clearly holds as  $2^0 = 1$ .

Now, if  $x_n = 2^{n-1}$  for some  $n \geq 1$ , then by definition,

$$x_{n+1} = 1 + 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 1 + 2^n - 1 = 2^n.$$

and hence the formula is valid for all  $n \geq 1$ .

44. We have

$$\begin{aligned}
 x_0 &= 0 \\
 x_1 &= 3/2 \\
 x_2 &= (3 - 3/2)/2 = 3/2 - 3/4 \\
 x_3 &= (3 - (3/2 - 3/4))/2 = 3/2 - 3/4 + 3/8 \\
 x_4 &= (3 - (3/2 - 3/4 + 3/8))/2 = 3/2 - 3/4 + 3/8 - 3/16 \\
 &\vdots \\
 x_n &= 3/2 - 3/4 - 3/8 + \cdots + 3/(-2)^n \\
 &= \frac{3}{2} \left( 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^3 + \cdots + \left(-\frac{1}{2}\right)^{n-1} \right) \\
 &= \frac{3(1/2)^n - 1}{2(-1/2) - 1} = - \left( \left(-\frac{1}{2}\right)^n - 1 \right).
 \end{aligned}$$

Now, we must prove it by induction. For  $n = 0$  the formula clearly holds as  $0 = -(1-1)$ .

Now, if  $x_n = -\left(\left(-1/2\right)^n - 1\right)$  for some  $n \geq 0$ , then by definition,

$$x_{n+1} = \left( 3 + \left( \left(-\frac{1}{2}\right)^n - 1 \right) \right) \cdot \frac{1}{2} = \left(-\frac{1}{2}\right)^n \frac{1}{2} + 1 = - \left( \left(-\frac{1}{2}\right)^{n+1} - 1 \right)$$

and hence the formula is valid for all  $n \geq 0$ .

45. (a) Since  $y^2 = y + 1$ , we can multiply this equation by  $y^n$  for any positive integer  $n$  to obtain  $y^{n+2} = y^{n+1} + y^n$ .

(b) By the quadratic formula,  $y = \frac{1 \pm \sqrt{5}}{2}$ .

(c) Let  $y_1 = \frac{1 + \sqrt{5}}{2}$  and  $y_2 = \frac{1 - \sqrt{5}}{2}$ , i.e., the two roots from (b).

For  $n = 1$ , we have

$$\frac{1}{\sqrt{5}}(y_1 - y_2) = \frac{\sqrt{5}}{\sqrt{5}} = 1 = x_1.$$

For  $n = 2$  [note that the case  $n = 2$  is necessary here!!!!], we have

$$\begin{aligned}
 \frac{1}{\sqrt{5}}(y_1^2 - y_2^2) &= \frac{1}{\sqrt{5}}((y_1 + 1) - (y_2 + 1)) && \text{[using (a)]} \\
 &= \frac{1}{\sqrt{5}}(y_1 - y_2) = 1 = x_2.
 \end{aligned}$$

So, now assume that

$$x_k = \frac{1}{\sqrt{5}}(y_1^k - y_2^k),$$

for all  $k$  such that  $1 \leq k \leq n$ , for some  $n \geq 2$ . [We must show that  $x_{n+1} = 1/\sqrt{5}(y_1^{n+1} - y_2^{n+1})$ .] Since  $n \geq 2$ , we have

$$\begin{aligned} \frac{1}{\sqrt{5}}(y_1^{n+1} - y_2^{n+1}) &= \frac{1}{\sqrt{5}}((y_1^n + y_1^{n-1}) - (y_2^n + y_2^{n-1})) && \text{[from (a)]} \\ &= \frac{1}{\sqrt{5}}(y_1^n - y_2^n) + \frac{1}{\sqrt{5}}(y_1^{n-1} - y_2^{n-1}) \\ &= x_n + x_{n-1} && \text{[by the ind. hyp.]} \\ &= x_{n+1} && \text{[by the recur. rel.]} \end{aligned}$$

## Homework 11

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46. Just use the definition.

47. (a)

$$(2 - x)^4 = 16 - 32x + 24x^2 - 8x^3 + x^4.$$

(b) The term with  $x^4$  is,

$$\binom{4}{2} (2x)^4 (-3)^{7-4} = -15120x^4.$$

So, the coefficient is  $-15120$ .

(c) The term is  $x^9$  is

$$\binom{20}{11} 5^1 1 (-7x)^9.$$

Since 9 is odd, the coefficient is negative.

48. We have:

$$2^n = (1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k}.$$

49. We have:

$$0 = 0^n = (-1 + 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k}.$$

50. We have that

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = a^n + na^{n-1}b + \sum_{k=0}^{n-2} \binom{n}{k} a^k b^{n-k}.$$

[Note that if  $n = 1$ , the sum ranges from 0 to  $-1$ , i.e., there is no term to be summed, and by convention this “empty sum” is defined to be zero. Observe that the formula above is still correct in this case.]

Now, since  $a, b \geq 0$ , we have that  $\binom{n}{k} a^k b^{n-k} \geq 0$ . Then, if  $S = \sum_{k=0}^{n-2} \binom{n}{k} a^k b^{n-k}$ , we have that  $S \geq 0$ . Hence,  $(a+b)^n = a^n + na^{n-1}b + S \geq a^n + na^{n-1}b$  [as we are subtracting a positive term].

52. Note that

$$\frac{n(n-1)(n-2)}{6} = \binom{n}{3}.$$

Then, if  $n \geq 3$ , we have [from Problem 48]

$$2^n = 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} + \sum_{k=4}^n \binom{n}{k} > \frac{n(n-1)(n-2)}{6}.$$

[Again, if  $n = 3$  the summation would be zero by definition.]

For  $n = 1$  and  $n = 2$ , we have that  $(n(n-1)(n-2))/6 = 0$ , while  $2^n > 0$ . Hence, we always have

$$2^n > \frac{n(n-1)(n-2)}{6}.$$

## Homework 12

### Pg. 60

1. Suppose that  $ab = 0$  and  $a \neq 0$ . Then, we have a multiplicative inverse, say  $c$ , such that  $ca = 1$ . Then,  $c(ab) = c \cdot 0 = 0$ , where the last equality comes from Example 5.5 [also done in class]. Since we have associativity, we have  $c(ab) = (ca)b = 0$ . And since  $c$  is the multiplicative inverse of  $a$ , we have  $1 \cdot b = 0$ . Since 1 is the multiplicative identity, we have  $1 \cdot b = b = 0$ .
2. Suppose that  $a \cdot b = a$  for all  $a$  in  $\mathbb{R}$  [on in an abstract field  $F$ ]. Then,  $1 \cdot b = 1$ . On the other hand, since 1 is the multiplicative identity, we must also have that  $1 \cdot b = b$ . Thus, putting these two inequalities together, we have  $1 = 1 \cdot b = b$ .

3. Suppose that  $ab = 1$  [and  $aq(a) = 1$ , by assumption; note that in class I used  $a^{-1}$  instead of  $q(a)$ ]. Then, we have that  $q(a) \cdot (ab) = q(a) \cdot 1 = q(a)$ . Also, we have  $(q(a) \cdot a)b = 1 \cdot b = b$ . Putting these two equations together, we have  $b = q(a)$ .
4. We have that  $q(q(a))$  is the multiplicative inverse of  $q(a)$ , i.e.,  $q(q(a))$  is the unique element such that  $q(q(a)) \cdot q(a) = 1$ . But since  $a \cdot q(a) = 1$  [as  $q(a)$  is the multiplicative inverse of  $a$ ], we then have that  $q(q(a)) = a$ .

We have that  $q(a)q(b)$  is the multiplicative inverse of  $ab$ , i.e.,  $q(a)q(b)$  is the unique element such that  $(q(a)q(b)) \cdot (ab) = 1$ . But since  $a \cdot q(a) = 1$  and  $b \cdot q(b) = 1$ , and since fields are commutative and associative, we have  $q(a)q(b)ab = (q(a)a)(q(b)b) = 1 \cdot 1 = 1$ .

5. We have that  $n(0)$  [which in class I'd denote by  $-0$ ], is the unique element such that  $n(0) + 0 = 0$ . But, since  $0$  is the additive identity, we have that  $0 + 0 = 0$ , and hence  $n(0) = 0$ .

We proved the other part in class. [Check your notes.]

6. As we've been doing above, it suffices to prove that  $(b + n(a)) + (a + n(b)) = 0$ . But, since we have associativity and commutativity, we have  $(b + n(a)) + (a + n(b)) = (b + n(b)) + (a + n(a)) = 0 + 0 = 0$ .
7. We have, by Problem 5,  $n(1) \cdot n(1) = n(n(1))$ , and by Example 5.7 [also done in class], we have that  $n(n(1)) = 1$ .
8. Since  $a \neq 0$ , we have the multiplicative inverse  $q(a)$  [or  $a^{-1}$ ]. Then,  $q(a)(ab) = q(a)(ac)$ . Since we have associativity, we have  $(q(a) \cdot a)b = (q(a) \cdot a)c$ , and hence  $1 \cdot b = 1 \cdot c$ , i.e.,  $b = c$ .

## Pg. 64

10. We have that if  $a$  is positive, then  $a > 0$  by definition. By Theorem 5.11 [also done in class], we have that  $-a < 0$ , and hence  $-a$  is negative.

In the same way, if  $a$  is negative, then  $a < 0$  by definition. By Theorem 5.11 again, we have that  $-a > 0$ , and hence  $-a$  is positive.

11. By Problem 10, we have that  $-c > 0$ . Then, since the inequality is multiplicative, we have that  $-ca < -cb$ . Then, since a field is commutativity, Theorem 5.11 tells us that  $ac > bc$ .

12. (a) We have that  $a + c < b + c$  [by adding  $c$  to  $a < b$ ], and  $b + c < b + d$  [by adding  $b$  to  $c < d$ ]. Hence, the transitive property of the inequality gives us that  $a + c < b + d$ .
- (b) We have that  $ac < bc$  [since  $c > 0$  and by multiplying by  $c$  the inequality  $a < b$ ], and  $bc < bd$  [since  $b > 0$ , as inequalities are transitive and  $a > 0$  and  $b > a$ , and by multiplying by  $b$  the inequality  $c < d$ ]. Hence, the transitive property of the inequality gives us that  $ac < bd$ .

15. We prove it by induction on  $n$ . For  $n = 1$ , we have that since  $a > 0$  and  $a < 1$ , that  $a \cdot a < 1 \cdot a$ , i.e.,  $a^2 < a$ .

Now, assume that  $a^{n+1} < a^n$ . [We need to prove that  $a^{n+2} < a^{n+1}$ .] Since  $a > 0$ , and the inequality is multiplicative, we get  $a \cdot a^{n+1} < a \cdot a^n$ , which, by definition of powers is the same as  $a^{n+2} < a^{n+1}$ .

17. (a) Before we proceed, we need the following result: if  $a > 0$ , then  $a^n > 0$  for all positive integers  $n$ . We prove it by induction on  $n$ . For  $n = 1$ , we have that  $a^1 = a > 0$  by assumption.

If now we assume  $0 < a^n$ , then since  $0 < a$ , Problem 12(b) tells us that  $0 \cdot 0 < a^n \cdot a$ , i.e.,  $0 < a^{n+1}$ .

We now prove the main result by induction on  $n$ . For  $n = 1$ , we have  $a^1 = a < b = b^1$  by assumption [and definition of powers].

Now, assume  $a^n < b^n$  for some  $n$ . By the first part of this solution, since  $a > 0$ , we have  $a^n > 0$ . Then, since  $0 < a < b$ , Problem 12(b) gives us that  $a^n \cdot a < b^n \cdot b$ , i.e.,  $a^{n+1} < b^{n+1}$ .

- (b) Suppose that  $a < b$ . [We are still assuming that  $a > 0$ .] Then, by the transitive property of inequalities, we have that  $b > 0$ . Also, we have  $a^n < b^n$  for  $n = 1$ .

So, suppose now that  $b > 0$  and  $a^n < b^n$  for some  $n$ . If  $a = b$ , then clearly  $a^n = b^n$  for all  $n$ , and hence  $a^n < b^n$  could not occur. Thus, we cannot have  $a = b$ . If  $b < a$ , then by (a) [with  $a$  and  $b$  switched] we would have that  $b^n < a^n$  for all  $n$ , and hence we would not be able to have  $a^n < b^n$ . Hence,  $b < a$  cannot occur either. Thus, by the trichotomy property of inequalities, we must have that  $a < b$ .

25. Suppose that  $a > 0$  and  $b < 0$ . Then, by Problem 10, we have  $0 < -b$ . Hence, since inequalities are multiplicative [and  $a > 0$ ], we have that  $0 \cdot a < -b \cdot a$ , i.e.,  $0 < -ab$ .

[Note that we have that  $-b \cdot a = (-1 \cdot b) \cdot a = -1 \cdot (a \cdot b) = -(a \cdot b)$ , using associativity and Problem 5.] Then, by Problem 10,  $-(-ab) = ab < 0$  [using Example 5.7].

If  $a < 0$  and  $b > 0$ , we can repeat the same proof above switching  $a$  and  $b$  to obtain that also  $ab < 0$ .

If either  $a$  or  $b$  is 0, then Example 5.5 tells us that  $ab = 0$ .

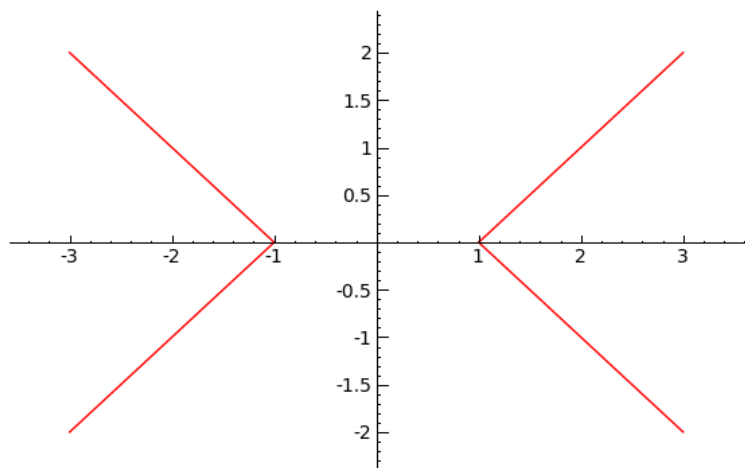
Therefore, if  $ab > 0$ , the only possibilities left [by trichotomy] are that both  $a$  and  $b$  are positive, or both are negative.

- 27.** Suppose first that  $a \leq b$ . Then, for all  $\epsilon > 0$ , we have that  $a + 0 < b + \epsilon$  by Problem 12(a) [if  $a < b$ ] or by the additive property of inequalities [if  $a = b$ , by adding  $a = b$  to  $0 < \epsilon$ ], i.e.,  $a < b + \epsilon$ .

Now assume that  $a < b + \epsilon$  for all  $\epsilon > 0$ . If  $a > b$ , then  $a - b > 0$  [by adding  $-b$  to  $a > b$ ]. Since  $1/2 < 1$ , we have that  $(a - b)/2 < (a - b)$  [by multiplying  $1/2 < 1$  by  $(a - b)$ ], and since  $1/2 > 0$ , we have that  $0 < (a - b)/2$  [by multiplying  $0 < 1/2$  by  $(a - b)$  and using Example 5.5]. So we can take  $\epsilon = (a - b)/2$ , and then,  $b + \epsilon = b + (a - b)/2 < b + (a - b)$  [by adding  $b$  to  $(a - b)/2 < (a - b)$ ], and hence  $b + \epsilon < a$ , which is a contradiction. Thus, by trichotomy,  $a \leq b$  [since  $a > b$  cannot hold].

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- 29.** (a) The graphs of the two parabolas  $y = x^2$  and  $y = -x^2$ .
- (c) We have  $|y| = 1 - |x|$ . Since  $|y| \geq 0$ , we must have  $|x| \leq 1$ , i.e.,  $-1 \leq x \leq 1$ . Then, we have  $y = \pm(1 - |x|)$ . [We know that graph of  $|x|$ , so graph transformations allows us to figure out  $\pm(1 - |x|)$ .] The graph then is a square with vertices  $(-1, 0)$ ,  $(1, 0)$ ,  $(-1, 1)$ , and  $(1, 1)$ .
- (d) We have that  $|y| = |x| - 1$  [and hence, since  $|y|$  must be greater than or equal to zero, we must have  $|x| \geq 1$ , i.e.,  $x \geq 1$  or  $x \leq -1$ ]. So, if  $y \geq 0$ , we have  $y = |x| - 1$ , and if  $y < 0$ , we have  $y = -(|x| - 1)$ . We have:



**30.** We have, by the triangle inequality, that

$$|a| = |(a - b) + b| \leq |a - b| + |b|,$$

and subtracting  $|b|$ , we obtain  $|a| - |b| \leq |a - b|$ .

**33.** [“ $\Rightarrow$ ”:] [We use the contrapositive.] Suppose that  $ab < 0$ . Then, exactly one of  $a$  and  $b$  is negative, while the other is positive. Suppose then  $a > 0$  and  $b < 0$ . Then,  $|b| = -b$ ,  $|a|, b < -b$ , and  $-a < a$ . Thus,  $|a| + |b| = a - b$  and  $|a + b|$  is either  $a + b$  or  $-a - b$ . Adding  $a$  to  $b < -b$  gives us  $a + b < a - b = |a| + |b|$ , while adding  $-b$  to  $-a < a$ , gives us  $-a - b < a - b = |a| + |b|$ . So, we must have that  $|a + b| < |a| + |b|$ .

If we have the other way around, i.e.,  $a < 0$  and  $b > 0$ , the same proof works by switching  $a$  and  $b$ .

[“ $\Leftarrow$ ”:] Assume that  $ab \geq 0$ .

If either  $a$  or  $b$  is zero, the result is trivial, as  $|x + 0| = |x| = |x| + |0|$ .

So, assume that  $ab > 0$ . Then either both  $a$  and  $b$  are positive or both are negative. Suppose that  $a, b > 0$ . Then,  $a + b > 0$  [since the order is additive], and hence  $|a + b| = a + b$ . Since  $a, b > 0$ , we also have  $|a| = a$  and  $|b| = b$ . Thus,  $|a + b| = a + b = |a| + |b|$ .

Suppose then that  $a, b < 0$ . Then,  $a + b < 0$  [since the order is additive], and hence  $|a + b| = -(a + b) = -a - b$ . Since  $a, b < 0$ , we also have  $|a| = -a$  and  $|b| = -b$ . Thus,  $|a + b| = -a - b = -a + (-b) = |a| + |b|$ .

**35.** We have

$$\begin{aligned} |x^2 - 1| &= |(x - 1)(x + 1)| \\ &= |x - 1| |x + 1| && \text{[by Prob. 32, done in class]} \\ &\leq |x - 1| (|x| + |1|) && \text{[by triang. ineq.]} \\ &\leq |x - 1| (1 + 1) && \text{[as } |x| \leq 1] \\ &= 2|x - 1|. \end{aligned}$$

**41.** By Theorem 5.14, we have that squares are always positive or zero. Hence, for all  $x, y \in \mathbb{R}$ , we have  $(x + y)^2 \geq 0$  and  $(x - y)^2 \geq 0$ . Hence, the former gives us that  $x^2 + y^2 \geq 2xy$  while the latter gives us that  $x^2 + y^2 \geq -2xy$ . Since  $|xy|$  is either  $xy$  or  $-xy$ , the inequalities above give us that  $x^2 + y^2 \geq 2|xy|$ .

**43.** By hypothesis, we have that  $|f(x_1)| \leq M$  and  $|f(x_2)| \leq M$  for all  $x_1, x_2 \in A$ . Note that  $|f(x_2)| = |-f(x_2)|$ . [This needs to be proved: if  $a \geq 0$ , then  $|a| = a$ , and  $-a \leq 0$ , and hence  $|-a| = -a = a$ . If  $a < 0$ , then  $-a > 0$ , and using the above, we have that  $|-a| = | -(-a) | = |a|$ . ]

Now Theorem 5.18, those give us that  $-M \leq f(x_1) \leq M$  and  $-M \leq -f(x_2) \leq M$  [using  $|f(x_2)| = |-f(x_2)| \leq M$ ]. Adding these two inequalities [which we can do by Problem 12(a)], we have

$$-2M \leq f(x_1) - f(x_2) \leq 2M.$$