Final (Take-Home Part)

M551 – Abstract Algebra

December 3rd, 2007

Due Date: Friday (12/08) by 10am. (If I am not in my office, please slide it under my door. *Do not put it in my mailbox!*)

You should not discuss *anything* about these problems with *anyone* [except me]. You can, however, use your book and notes. Feel free to come talk to me [or write me] if you have questions.
Since you have some time, please make your solutions *neat* and well written. Please try to finish by the deadline, but if you feel you need more time, please let me know ASAP. [I will not extend it for later than Monday (12/10), though.]

1. Let R be a commutative ring with identity. Suppose that for each prime ideal P, the localization R_P has no non-zero nilpotent element.

(a) [8 points] Show that R has no non-zero nilpotent element.

(b) [7 points] Is R necessarily a domain?

2. Let R be a non-Noetherian commutative ring with identity, and $\mathcal{S}$ be the set of ideals which are *not* finitely generated.

(a) [5 points] Show that $\mathcal{S}$ has a maximal element I [with respect to the inclusion]. [The ideal I in the next items is this maximal element.]

(b) [8 points] Suppose that $x \notin I$. Prove that there exists a *finitely generated* ideal $I_0 \subseteq I$, such that $(I_0, x) = (I, x)$. [Don't forget the $I_0 \subseteq I$ part!]

(c) [4 points] Suppose $xy \in I$, but $x, y \notin I$. Prove that $J \stackrel{\text{def}}{=} \{r \in R \ : \ rx \in I\}$ is a finitely generated ideal.

(d) [8 points] Prove that I must be prime. [Of course, use (b) and (c). Assume that I is not prime and conclude that it must be finitely generated.]

[Note that this proves that if every prime ideal of a commutative ring with 1 is finitely generated, then the ring is Noetherian.]