Midterm

M551 – Abstract Algebra

October 18th, 2007

Solve as many as you can in class. Then, take this sheet home and solve all remaining problems, or problems that you think you've missed, at home and bring it to class on Monday.

You should treat these problem as a *take-home* exam, not as a homework. So, you should not discuss *anything* about these problems with *anyone*. You can, however, use your book and notes.

1. Let $H \triangleleft G$.

(a) Show that if G is finite and G/H has an element of order n, for some positive integer n, then G also has an element of order n.

(b) Show that the conclusion of part (a) doesn't always hold if G is infinite.

2. Let $H \leq \mathrm{Aut}(N)$, and assume that no non-identity element of H fixes *any* non-identity element of N. [I.e., if $h \neq 1$ and $n \neq 1$, then $h(n) \neq n$.] Let $G \stackrel{\text{def}}{=} N \rtimes H$ and identify N and H with the corresponding subgroups of G.

(a) Show that $H \cap gHg^{-1} = 1$ for all $g \in G - H$.

(b) If G is finite, show that $G = N \cup \left(\bigcup_{g \in G} gHg^{-1} \right)$.

3. Prove that if G is nilpotent [possibly *infinite*], and $H < G$, then $H < N_G(H)$.

4. Let G be a group with $|G| = p(p+1)$, where $p > 2$ is prime. Assume that G has no normal Sylow p-subgroup.

(a) Let $P \in \mathrm{Syl}_p(G)$, $|x| \neq 1, p$, and $S \stackrel{\text{def}}{=} \{1\} \cup \{yxy^{-1} \ : \ y \in P\}$. Prove that $|S| = p+1$, and if $z \in S$, then $z^2 = 1$.

(b) Prove that $(p+1) = 2^r$ for some positive integer r, and that G has a normal subgroup of order $(p+1)$.