1) Answer all giving short explanations.

(a) Let \( V \) be a vector space and \( v \in V \). When is \( \{v\} \) linearly independent? [No need to explain this one.]

*Solution.* Remember, a single vector is linearly independent if, and only if, this vector is non-zero. [Since a linear combination of the single vector is \( kv \). If \( kv = 0 \), then either \( k = 0 \) or \( v = 0 \). So, it's linearly independent if, and only if, \( v \neq 0 \).]

(b) Is the set \( \{(1, 2, 3, 4, 5), (-2, -4, -6, -8, -9)\} \) linearly independent [in \( \mathbb{R}^5 \)]?

*Solution.* Yes, since one is not a multiple of the other.

(c) Is the set \( \{(-5, \sqrt{2}), (\pi, e), (\ln(3), 1/2)\} \) linearly independent [in \( \mathbb{R}^2 \)]?

*Solution.* No, since \( \text{dim} \mathbb{R}^2 = 2 \) and we have 3 vectors. [More vectors than the dimension always gives us linearly dependent sets.]

(d) Does the set \( \{1 + x + x^3, -2 + x^2, 1 + x - x^2 + x^3\} \) span all of \( P_3 \) [i.e., all polynomials of degree less than or equal to 3]?

*Solution.* No, since \( \text{dim} P_3 = 4 \) and we only have 3 vectors. [Less vectors than the dimension of the space cannot span the space.]
2) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator for which

$$
T \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = 
\begin{pmatrix}
2 \\
-1 \\
0
\end{pmatrix},
T \begin{pmatrix}
0 \\
2 \\
0
\end{pmatrix} = 
\begin{pmatrix}
2 \\
-2 \\
6
\end{pmatrix},
T \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix} = 
\begin{pmatrix}
3 \\
2 \\
1
\end{pmatrix}.
$$

(a) Find the matrix $[T]$ associated to the linear transformation $T$.

\textit{Solution.} We have

\[
[T] = \begin{bmatrix}
T(e_1) & T(e_2) & T(e_3)
\end{bmatrix}
\]

But $T(2e_2) = (2, -2, 6)$, and since $T$ is linear, we have that $T(2e_2) = 2T(e_2)$. So, $T(e_2) = (1, -1, 3)$.

We have that $T(e_1 + e_3) = (3, 2, 1)$. But since $T$ is linear, we have that $T(e_1 + e_3) = T(e_1) + T(e_3)$. So, $T(e_3) = (3, 2, 1) - (2, -1, 0) = (1, 3, 1)$.

Thus,

\[
[T] = \begin{bmatrix}
2 & 1 & 1 \\
-1 & -1 & 3 \\
0 & 3 & 1
\end{bmatrix}
\]

\[
\]

(b) Is $T$ one-to-one? Is it onto? [Don’t forget to justify!!]

\textit{Solution.} We have that $\det[T] = -22 \neq 0$, we have that $T$ is both onto and one-to-one.

\[
\]
3) Let $T: \mathbb{R}^m \to \mathbb{R}^n$ and $W$ be the range of $T$. In other words, the elements of $W$ are of the form $T(v)$, where $v \in \mathbb{R}^m$. Prove that $W$ is a vector space.

Solution. Since $W \subseteq \mathbb{R}^n$ [with the same addition and scalar multiplication], we just need to show it is a subspace of $\mathbb{R}^n$, which is a lot simpler.

[Note that $W$ is not empty, since, for instance, $T(0) = 0 \in W$.]

Two elements of $W$ are of the form $T(v)$ and $T(w)$ where $v, w \in \mathbb{R}^m$. Then, since $T$ is linear, $T(v) + T(w) = T(v + w)$. Since $v + w \in \mathbb{R}^m$, we have that $T(v) + T(w) = T(v + w) \in W$. [So it’s closed under addition.]

Now, if $k \in \mathbb{R}$, then, since $T$ is linear, $kT(v) = T(kv)$. Since $kv \in \mathbb{R}^n$, we have that $kT(v) = T(kv) \in W$. [So it’s closed under scalar multiplication.]

So, since $W$ in a non-empty subset of $\mathbb{R}^m$ which is closed under addition and scalar multiplication, we have that $W$ is a subspace of $\mathbb{R}^m$ and hence it is itself a vector space.

Here is another solution. Let $[T]$ be the matrix associated to $T$. Then the range of $T$ is the set of vectors $T(x) = [T] \cdot x$ such that $x \in \mathbb{R}^m$. But, if $[T] = [c_1 \cdots c_m]$ [i.e., the $c_i$’s are the columns of $[T]$], then

$$[T] \cdot x = [c_1 \cdots c_m] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 c_1 + \cdots x_m c_m,$$

where $x_1, \ldots, x_m \in \mathbb{R}$. Hence, the range of $T$ is span$\{c_1, \ldots, c_m\}$, i.e., the column space of $[T]$. Therefore, it’s a vector space [in fact, a subspace of $\mathbb{R}^n$].

$\square$
4) Let

\[
A = \begin{bmatrix}
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & -4 \\
1 & 0 & -1 & 0 & 2 \\
2 & 2 & 2 & 0 & 4 \\
\end{bmatrix}
\]

(a) Find bases for the nullspace, column space, and row space of \(A\), with the requirement that the basis for the column space of \(A\) is composed of columns of \(A\). [There is no requirement for the row and null spaces.]

**Solution.** Putting \(A\) in reduced echelon form, we get:

\[
\begin{bmatrix}
1 & 0 & -1 & 0 & 2 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So, the basis for the row space of \(A\) is

\[
S_{\text{row}} = \{(1, 0, -1, 0, 2), (0, 1, 2, 0, 0), (0, 0, 0, 1, -2)\}.
\]

Since the first, second, and fourth columns have the leading ones, we get that a basis for the column space \(A\) [made of columns of \(A\)] is:

\[
S_{\text{col}} = \{(0, 0, 1, 2), (1, 0, 0, 2), (0, 2, 0, 0)\}.
\]

For the nullspace, note that the general solution of \(A\mathbf{x} = \mathbf{b}\) is [from the echelon form]:

\[
\mathbf{x} = \begin{bmatrix} t - 2s \\ -2t \\ t \\ 2s \\ s \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}.
\]

So, the basis for the nullspace of \(A\) is

\[
S_{\text{null}} = \{(1, -2, 1, 0, 0), (-2, 0, 0, 2, 1)\}.
\]
(b) Let $S$ be the basis for the column space that you’ve found in (a). Then, for each column $c_i$ of $A$, find $(c_i)_S$ [i.e., write the coordinate vector of this column with respect to the basis $S$].

**Solution.** Let $c'_i$ denote the columns of the echelon form. Then, we can easily see that

$$c'_3 = -c'_1 + 2c'_2 + 0c'_4 \quad \text{and} \quad c'_5 = 2c'_1 + 0c'_2 - 2c'_4.$$ 

So, we have:

$$c_3 = -c_1 + 2c_2 + 0c_4 \quad \text{and} \quad c_5 = 2c_1 + 0c_2 - 2c_4.$$

Since, $S = \{c_1, c_2, c_4\}$, we have $(c_3)_S = (1, 2, 0)$ and $(c_5) = (2, 0, -2)$. Also, clearly $(c_1)_s = (1, 0, 0)$, $(c_2)_S = (0, 1, 0)$, and $(c_4)_S = (0, 0, 1)$. 

\[\square\]