1) Suppose that $|G| = 2p$, where $p$ is a prime different from 2. Prove that either $G \cong C_{2p}$ or $G \cong D_{2p}$.

Proof. By the First Sylow Theorem, [since 2 and $p$ are both primes and $p \neq 2$] there are subgroups $H$ and $K$ such that $|H| = p$ and $|K| = 2$. Hence, since they have prime orders, $H \cong C_p$ and $K \cong C_2$. Let $H = \langle x \rangle$ and $K = \langle y \rangle$.

Since $[G : H] = 2$, we have that $H \triangleleft G$. [We could also obtain that from Third Sylow Theorem.] We also have that $H \cap K = \{1\}$ [since their orders are relatively prime], and, by Proposition 2.8.6(a), since $|H| \cdot |K| = |G|$, we have $H \cdot K = G$. Therefore,

$$G = \{1, x, x^2, \ldots, x^{p-1}, y, xy, x^2y, \ldots, x^{p-1}y\}.$$ 

If $K \triangleleft G$, then we have, by Proposition 2.8.6(c), that $G \cong H \times K \cong C_p \times C_2 \cong C_{2p}$. [In the last equality, we used the fact that $p \neq 2$.]

Suppose then that $K$ is not normal. By the Second Sylow Theorem, we have that there is more than one Sylow 2-subgroup, while there is only one Sylow $p$-subgroup [namely, $H$]. By the Third Sylow Theorem, $s_2$ [i.e., the number of Sylow 2-subgroups] divides $p$, so it is either 1 or $p$. Since it is not 1 [as we’ve seen above], it must be $p$. So, we have $p$ elements of orders 2. Since all $p$ elements of $H$ do not have order 2 [they have order $p$ or 1], all other elements must have order 2. So, $y, xy, \ldots, x^{p-1}y$ all have order two. So, $xy$ has order two, and:

$$(xy)^2 = xyyx = 1 \quad \Rightarrow \quad yx = x^{-1}y^{-1} = x^{-1}y$$

[since $y$ also has order two]. Thus, $G = \langle x, y \rangle$, $x$ has order $p$, $y$ has order 2, and $yx = x^{-1}y$. Therefore $G \cong D_{2p}$. 

$\square$
2) Let $H \triangleleft G$, $\bar{K} < G/H$, and

$$K \overset{\text{def}}{=} \{ x \in G : x \in gH \text{ for some } gH \in \bar{K} \}$$

[i.e., $K$ is the union of all cosets in $\bar{K}$].

(a) Prove that $K$ is a subgroup of $G$ containing $H$.

Solution. Let $x, y \in K$. So, [by defn. of $K$] there are $g_1 H, g_2 H \in \bar{K}$ such that $x \in g_1 H$ and $y \in g_2 H$. Thus, $y^{-1} \in H g_2^{-1} = g_2^{-1} H = (g_2 H)^{-1}$ [since $H \triangleleft G$ and $\bar{K} < G/H$]. Therefore $xy^{-1} \in (g_1 H)(g_2 H)^{-1}$. Since $\bar{K} < G/H$, we have that $(g_1 H)(g_2 H)^{-1} = (g_1 g_2^{-1})H \in \bar{K}$. Hence, $xy^{-1} \in K$. By the one-step method, $K < G$.

Now, since $1 \cdot H = H \in \bar{K}$, all its elements are in $K$. \hfill \Box

(b) Prove that $\bar{K} = \{ kH : k \in K \}$.

Solution. Let $gH \in \bar{K}$. Then $g \cdot 1 = g \in K$. Therefore, $gH \in \{ kH : k \in K \}$, and $\bar{K} \subseteq \{ kH : k \in K \}$.

Let $k \in K$. Then $k \in gH$ for some $gH \in \bar{K}$. So, $kH = gH$ [since the cosets are disjoint]. Hence, $kH \in \bar{K}$, and $\{ kH : k \in K \} \subseteq \bar{K}$.

Thus, $\bar{K} = \{ kH : k \in K \}$. \hfill \Box
3) Let $M, N \triangleleft G$.

(a) Prove that $(NM) < G$, $M \triangleleft (NM)$, and $(N \cap M) \triangleleft N$.

Solution. Since, $M, N \triangleleft G$, by Proposition 2.8.6(b), $NM < G$.
Let $m \in M$ and $g \in NM$. Since $NM \subseteq G$ and $M \triangleleft G$, $gmg^{-1} \in M$, and so $M \triangleleft NM$.
We will prove that $(N \cap M) \triangleleft N$ in (b) below. [Or, you can just quote Proposition 2.7.1.]

(b) Prove that $N/(N \cap M) \cong (NM)/M$.

Solution. Let $\phi : N \rightarrow (NM)/M$ defined by $\phi(n) = nM$. [Note that since $N \subseteq NM$, we have $nM \in (NM)/M$.]
We have $\phi(n_1n_2) = (n_1n_2)M = (n_1M)(n_2M)$, and hence $\phi$ is a homomorphism.
Given $nmM \in (NM)/M$, we have that $nmM = nM$, since $nmM^{-1} = n \in nmM$ [and cosets are disjoint]. So, $\phi(n) = nM = nmM$, and $\phi$ is onto.
We have that $\phi(n) = M$ iff $nM = M$ iff $n \in M$. Since we also have that $n \in N$, we obtain $\ker \phi = N \cap M$. [In particular, this proves that $(N \cap M) \triangleleft N$ for part (a).]
By the First Isomorphism Theorem, $N/(N \cap M) \cong (NM)/M$. 

□
4) Let \( G \) be an Abelian group, \( H < G \) and \( \phi : G \to H \) be a homomorphism such that \( \phi(h) = h \) for all \( h \in H \). Prove that \( G \cong H \times \ker \phi. \) [Hint: Remember that \( \phi(g) = \phi(g') \) iff \( g^{-1}g' \in \ker \phi \).]

Solution. Yet again, we use Proposition 2.8.6.

\([H, \ker \phi \triangleleft G:]\) Since \( G \) is Abelian, both \( H \) and \( \ker \phi \) are normal subgroups of \( G \).

\([H \cap \ker \phi = \{1\}:]\) Let \( g \in H \cap \ker \phi \). In particular \( g \in H \), and so \( \phi(g) = g \). On the other hand, also \( g \in \ker \phi \), and so \( \phi(g) = 1 \). Thus, \( g = \phi(g) = 1 \), and \( H \cap \ker \phi = \{1\} \) [since we proved that an arbitrary element of \( H \cap \ker \phi \) has to be equal to 1].

\([H \cdot \ker \phi = G:]\) Let \( g \in G \). Then \( \phi(g) \in H \). So, denote \( h \overset{\text{def}}{=} \phi(g) \). Then, since \( h \in H \), we have that \( \phi(h) = h = \phi(g) \). By the hint, \( h^{-1}g \in \ker \phi \). But then, \( g = h \cdot (h^{-1}g) \in H \cdot \ker \phi \).

Since \( g \) was arbitrary, we have \( H \cdot \ker \phi = G \).

By Proposition 2.8.6(c), \( G \cong H \times \ker \phi \).
5) Let $R$ be a [not necessarily commutative] ring in which $a^2 = a$ for all $a \in R$.

(a) Prove that for all $a \in R$, we have $a = -a$.

Solution. We have $-a = (-a)^2 = (-a)(-a) = a^2 = a$. [Remember that it was proved in class that $(-x)(-y) = xy$.]

(b) Prove that $R$ is commutative. [Hint: Expand $(a + b)^2$ in the ring.]

Solution. We have

\[
(a + b)^2 = (a + b)(a + b) \\
= a(a + b) + b(a + b) \\
= a^2 + ab + ba + b^2 \\
= a + ab + ba + b.
\]

On the other hand, $(a + b)^2 = (a + b)$. So,

\[
a + ab + ba + b = a + b \quad \Rightarrow \quad ab + ba = 0 \\
\quad \Rightarrow \quad ab = -ba \\
\quad \Rightarrow \quad ab = ba.
\]

[where the last statement comes from part (a)].