Let $G \overset{\text{def}}{=} \mathbb{Z}/36\mathbb{Z}$ and $H \overset{\text{def}}{=} \langle \bar{2} \rangle \cap \langle \bar{3} \rangle$. [As usual, $\bar{a}$ represents the coset $(a + 36\mathbb{Z})$ of $\mathbb{Z}/36\mathbb{Z}$.]

(a) Describe $G/H$ as a set. [In other words, give its elements.]

**Solution.** We have that:

$$\langle \bar{2} \rangle = \{0, 2, 4, \ldots, 32\}$$
$$\langle \bar{3} \rangle = \{0, 3, 6, \ldots, 33\}$$

Thus,

$$\langle \bar{2} \rangle \cap \langle \bar{3} \rangle = \langle \bar{6} \rangle = \{0, 6, 12, \ldots, 30\},$$

and

$$G/H = \{H, (\bar{1} + H), (\bar{2} + H), (\bar{3} + H), (\bar{4} + H), (\bar{5} + H)\}.$$

(b) To what group is $G/H$ isomorphic? [Give a precise description, like $S_3$, $Q_8$, $C_7$, $C_2 \times C_2$, $\mathbb{Z}$, etc.]

**Solution.** Since $\mathbb{Z}/36\mathbb{Z}$ is cyclic [$\mathbb{Z}/36\mathbb{Z} = \langle \bar{1} \rangle$], we know that $G/H$ is cyclic. [We proved in class that every quotient group of a cyclic group is also cyclic.] Since $|G/H| = 6$, we have $G/H \cong C_6$.

[Even if you did not remember that, it would be easy to verify: just note that $G/H$ is Abelian (and so not isomorphic to $S_3$) or that $G/H = \langle \bar{1} + H \rangle$, since $(\bar{1} + H)$ has order 6.]
2) Let \( G \overset{\text{def}}{=} \mathbb{R}^\times \times \mathbb{R}^\times \) act on \( S \overset{\text{def}}{=} \mathbb{R}^2 \) by: given \((a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times\), and \((x, y) \in \mathbb{R}^2\),

\[ f_{(a,b)}(x, y) \overset{\text{def}}{=} (ax, by). \]

(a) Prove that this indeed defines a group action.

\textit{Proof.} (i) The identity of \( \mathbb{R}^\times \times \mathbb{R}^\times \) is \((1, 1)\) and

\[ f_{(1,1)}(x, y) = (x, y). \]

Hence \( f_{(1,1)} \) is the identity function.

(ii) Let \((a, b), (c, d) \in \mathbb{R}^\times \times \mathbb{R}^\times\). Then

\[
\begin{align*}
    f_{(a,b)} \circ f_{(c,d)}(x, y) &= f_{(a,b)}(f_{(c,d)}(x, y)) = f_{(a,b)}(cx, dy) \\
    &= (acx, bdy) = f_{(ac,bd)}(x, y) \\
    &= f_{(a,b)(c,d)}(x, y).
\end{align*}
\]

So, \( f_{(a,b)} \circ f_{(c,d)} = f_{(a,b)(c,d)} \).

(b) Describe the orbits of \((1, -3)\) and \((-\pi, 0)\) geometrically. [Like, “the circle of radius 3 and center at the origin”, or “the vertical line passing though \(-2\), or “the line \(x = y\) minus the point \((1, 1)\)”, etc.]

\textit{Solution.} We have:

\[
O_{(1,-3)} = \{ f_{(a,b)}(1, -3) : (a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times \}
\]

\[
= \{ (a, -3b) : (a, b) \}
\]

\[
= \{ (x, y) \in \mathbb{R}^2 : x, y \neq 0 \}.
\]

To see the last equality notice that: for all \((a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times\), \(a, -3b \neq 0\) [and hence we have “\(\subseteq\)’’], and for all \((x, y)\) with \(x, y \neq 0\), we have \(f_{(x,-x/3)}(1, -3) = (x, y)\) [and hence we have “\(\supset\)’’]. Thus, \(O_{(1,-3)}\) is the plane minus the \(x\) and \(y\)-axes.

Also:

\[
O_{(-\pi,0)} = \{ f_{(a,b)}(-\pi, 0) : (a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times \}
\]

\[
= \{ (-a\pi, 0) \cdot b) : (a, b) \}
\]

\[
= \{ (x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}^\times \}.
\]

To see the last equality notice that: for all \((a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times\), \(-a\pi \neq 0\) [and hence we have “\(\subseteq\)’’], and for all \((x, 0)\) with \(x \neq 0\), we have \(f_{(-x/\pi,1)}(-\pi, 0) = (x, 0)\) [and hence we have “\(\supset\)’’]. Thus, \(O_{(-\pi,0)}\) is the \(x\)-axis minus the origin.

\[ \square \]
(c) Describe the stabilizers of $(1, -3)$ and $(-\pi, 0)$.

*Solution.* We have:

\[
(R^x \times R^x)_{(1,-3)} = \{(a, b) \in R^x \times R^x) : f_{(a,b)}(1, -3) = (1, -3)\}
\]

\[
= \{(a, b) \in R^x \times R^x) : (a, -3b) = (1, -3)\}
\]

\[
= \{(1, 1)\}
\]

and

\[
(R^x \times R^x)_{(-\pi,0)} = \{(a, b) \in R^x \times R^x) : f_{(a,b)}(-\pi, 0) = (-\pi, 0)\}
\]

\[
= \{(a, b) \in R^x \times R^x) : (-a \pi, 0) = (-\pi, 0)\}
\]

\[
= \{(1, b) : b \in R^x\}
\]

\[
= \{1\} \times R^x.
\]

\[\square\]
3) Prove the following:

(a) Let $G$ be a finite group. Prove that for all $a \in G$, we have $a^{|G|} = 1_G$. [Note: You cannot use item (b) in this proof!]

**Proof.** Let $n$ be the order of $a$. Hence, $|\langle a \rangle| = n$, and $a^n = 1_G$. By Lagrange’s Theorem, $n$ divides $|G|$ [since $\langle a \rangle < G$]. Thus $|G| = n \cdot k$, for some integer $k$. So,

$$a^{|G|} = a^{nk} = (a^n)^k = 1^k_G = 1_G.$$

(b) Let $H \unlhd G$ with $[G : H] = n$. Prove that for all $a \in G$, we have $a^n \in H$. [Note: You can use item (a) in this proof, even if you didn’t do it.]

**Proof.** Let $a \in G$. Then, since $H \unlhd G$, we have a quotient group $G/H$ and $|G/H| = [G : H] = n$. Thus, by item (a), $(aH)^{|G/H|} = (aH)^n = H$ [since $H = 1H$ is the unit of $G/H$]. On the other hand $(aH)^n = a^nH$. Thus $a^nH = H$, and therefore, $a^n \in H$. 

\[\square\]
4) Let $G$ be an Abelian group and

$$\Delta \overset{\text{def}}{=} \{(g, g) : g \in G\}.$$ 

Prove that $\Delta \triangleleft G \times G$ and $(G \times G)/\Delta \cong G$.

**Proof.** [We will use the First Isomorphism Theorem.] Let

$$\phi : G \times G \to G$$

defined by $\phi(g, h) = gh^{-1}$. Then,

$$\phi((g_1, h_1)(g_2, h_2)) = \phi(g_1g_2, h_1h_2)$$  \hspace{1cm} \text{[prod. in $G \times G$]}

$$= (g_1g_2)(h_1h_2)^{-1}$$  \hspace{1cm} \text{[defn. of $\phi$]}

$$= g_1h_1^{-1}g_2h_2^{-1}$$  \hspace{1cm} \text{[G is Abelian]}

$$= \phi(g_1, h_1)\phi(g_2, h_2)$$  \hspace{1cm} \text{[defn. of $\phi$]},

and so $\phi$ is a homomorphism.

Now,

$$(g, h) \in \ker \phi \iff gh^{-1} = 1_G$$

$$\iff g = h$$

$$\iff (g, h) \in \Delta.$$

So, $\Delta = \ker \phi$. Therefore $\Delta \triangleleft G \times G$ [the kernel of a homomorphism is always a normal subgroup].

Moreover $\phi$ is onto, since given $g \in G$, $\phi(g, 1_G) = g$. Thus, by the First Isomorphism Theorem, $(G \times G)/\Delta \cong G$. 

$\square$