1) Let $G \triangleq C_4 \times C_8$. [As usual, $C_n$ denotes the cyclic group of order $n$.] Let $x$ and $y$ denote the generators of $C_4$ and $C_8$ respectively, i.e., $C_4 = \langle x \rangle$ and $C_8 = \langle y \rangle$, and let $H \triangleq \langle (x, y^7) \rangle$.

(a) Give the elements of $H$ explicitly.

Solution.

$$H = \langle (x, y^7) \rangle = \{(x, y^7)^k : k \in \mathbb{Z}\} = \{(1, 1), (x, y^7), (x^2, y^6), (x^3, y^5), (1, y^4), (x, y^3), (x^2, y^2), (x^3, y)\}.$$

(b) Describe $G/H$ as a set. [In other words, give its elements.]

Solution. We know that $|G| = 4 \cdot 8 = 32$ and $|H| = 8$. Thus, $|G/H| = |G|/|H| = 4$. [This makes our lives easier, since we now have only to find three cosets besides $H$ itself.] Since $(x, 1)$ is not in $H$, we have that $(x, 1)H \neq H$. We also have $(x^2, 1) \notin H, (x, 1)H$, so it gives another coset. Finally, since $(x^3, 1) \notin H, (x, 1)H, (x^2, 1)H$, we have that

$$G/H = \{H, (x, 1)H, (x^2, 1)H, (x^3, 1)H\}.$$

(c) To what group is $G/H$ isomorphic? [Give a precise description, like $S_3$, $Q_8$, $C_7$, $C_2 \times C_2, \mathbb{Z}$, etc.]

Solution. We have that

$$G/H = \langle (x, 1)H \rangle,$$

and hence $G/H \cong C_4$. 

\[ \square \]
2) Let \( G = (0, \infty) \times \mathbb{R} \) act on \( S \overset{\text{def}}{=} \mathbb{R}^2 \) by: given \((r, t) \in G \) and \((x, y) \in S\),

\[ f_{(r, t)}(x, y) \overset{\text{def}}{=} (rx, y + t). \]

(a) Prove that this indeed defines a group action.

*Solution.*

(i) The identity of \((0, \infty) \times \mathbb{R}\) is \((1, 0)\). Then:

\[ f_{(1, 0)}(x, y) = (1 \cdot x, y + 0) = (x, y). \]

Thus, \( f_{(1, 0)} \) is the identity function.

(ii) Given \((r_1, t_1), (r_2, t_2) \in (0, \infty) \times \mathbb{R}\), we have

\[
\begin{align*}
  f_{(r_1, t_1)} \circ f_{(r_2, t_2)}(x, y) &= f_{(r_1, t_1)}(r_2 x, y + t_2) \\
  &= (r_1 r_2 x, y + t_1 + t_2) \\
  &= f_{(r_1 r_2, t_1 + t_2)}(x, y) \\
  &= f_{(r_1, t_1)(r_2, t_2)}(x, y).
\end{align*}
\]

\( \square \)

(b) Describe the orbits of \((-\sqrt{2}, \pi)\) and \((0, 1)\).

*Solution.* We have:

\[
\begin{align*}
  O_{(-\sqrt{2}, \pi)} &= \{ f_{(r, t)}(-\sqrt{2}, \pi) : (r, s) \in (0, \infty) \times \mathbb{R} \} \\
  &= \{ (-r \sqrt{2}, \pi + t) : (r, s) \in (0, \infty) \times \mathbb{R} \} \\
  &= \{ (x, y) : x < 0 \}.
\end{align*}
\]

Hence, this orbit is the half plane on the left of the \( y \)-axis.

Also,

\[
\begin{align*}
  O_{(0, 1)} &= \{ f_{(r, t)}(0, 1) : (r, s) \in (0, \infty) \times \mathbb{R} \} \\
  &= \{ (0, 1 + t) : (r, s) \in (0, \infty) \times \mathbb{R} \} \\
  &= \{ (0, y) : y \in \mathbb{R} \}.
\end{align*}
\]

Hence, this orbit is the \( y \)-axis. \( \square \)
(c) Describe the stabilizers of \((-\sqrt{2}, \pi)\) and \((0, 1)\).

Solution. We have:

\[
G_{(-\sqrt{2}, \pi)} = \{(r, t) \in G : f_{(r,t)}(-\sqrt{2}, \pi) = (-\sqrt{2}, \pi)\} \\
= \{(r, t) \in G : (-r\sqrt{2}, \pi + t) = (-\sqrt{2}, \pi)\} \\
= \{(1, 0)\}.
\]

Also,

\[
G_{(0,1)} = \{(r, t) \in G : f_{(r,t)}(0, 1) = (0, 1)\} \\
= \{(r, t) \in G : (0, 1 + t) = (0, 1)\} \\
= (0, \infty) \times \{0\}.
\]
3) Let $G$ be a group with normal subgroups of orders 3 and 5. Prove that $G$ has an element of order 15.

[If you don’t think you can do this, you can try to do it with the assumption that $G$ is Abelian. It’s easier, but you will only get half of the credit.]

Solution. Let $H$ be the subgroup of order 3 and $K$ be the subgroup of order 5. Since $H \cap K$ is a subgroup of both $H$ and $K$, its order dividers both orders, i.e., it divides both 3 and 5. Hence, $|H \cap K| = 1$, i.e., $H \cap K = \{1_G\}$.

For $G$ Abelian: Since their orders are prime, they are both cyclic generated by any non-identity element. Let $x$ and $y$ be their respective generators.

We claim that $xy$ has order 15: since $G$ is Abelian, we have that $(xy)^k = x^ky^k$. Then $(xy)^{15} = x^{15}y^{15} = 1_G$. So, the order of $xy$ divides 15. But $(xy)^3 = x^3y^3 = y^3 \neq 1_G$ and $(xy)^5 = x^5y^5 = x^2 \neq 1_G$. Hence the order of $xy$ is indeed 15.

For $G$ not Abelian: Now, let us not assume that $G$ is Abelian, but that $H, K \triangleleft G$. By Proposition 2.8.6 from Artin’s text, we have that $HK \cong H \times K$. [Note that we don’t necessarily have that $HK = G$!!] But then, since $H \cong C_3$ and $K \cong C_5$ and gcd(3, 5) = 1, we have that $H \times K \cong C_{15}$ and hence it has an element of order 15. Therefore, so does $HK$ [and hence, since $HK \subseteq G$, so does $G$].

[In fact, if you look at the proof given in Proposition 2.8.6, you see that if $H, K \triangleleft G$ with $H \cap K = \{1_G\}$, then for all $h \in H$ and $k \in K$, we have $hk = kh$. (Note that this is not the same as $HK = KH$!!!) But then, you can also copy the proof for Abelian groups, since the generators will commute with each other!]
4) Let $G \overset{\text{def}}{=} \mathbb{Z} \times \mathbb{Z}$ and 

$$H \overset{\text{def}}{=} \{(n, -n) : n \in \mathbb{Z}\}.$$ 

Prove that $H \triangleleft G$ and $G/H \cong \mathbb{Z}$.

**Solution.** Let $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(n, m) = n + m$.

(i) $\phi$ is a homomorphism: Let $(n_1, m_1), (n_2, m_2) \in \mathbb{Z} \times \mathbb{Z}$. Then,

$$\phi((n_1, m_1) + (n_2, m_2)) = \phi(n_1 + n_2, m_1 + m_2)$$

$$= n_1 + n_2 + m_1 + m_2$$

$$= (n_1 + m_1) + (n_2 + m_2)$$

$$= \phi(n_1, m_1) + \phi(n_2, m_2).$$

(ii) $\ker \phi = H$: $\phi(n, m) = 0$ iff $n + m = 0$ iff $m = -n$ iff $(n, m) \in H$. This gives us also that $H \triangleleft G$.

(iii) $\phi$ is onto: given $n \in \mathbb{Z}$, we have $\phi(n, 0) = n$.

Therefore, by the *First Isomorphism Theorem*, we have that $G/H \cong \mathbb{Z}$. 

\[\square\]