Gravitational Self-Force

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In general relativity, it makes perfectly good sense to consider the motion of a body in a fixed spacetime (i.e., a test body) in a point particle limit. In this limit, geodesic motion is obtained. However, it does not make any sense to consider Einstein's equation with a "point mass" source for the metric; Einstein's equation with a point mass stress-energy does not make mathematical sense.

It also is of considerable interest to determine the motion of a "small" body in general relativity, taking into account the deviations from geodesic motion arising from gravitational self-force effects. One would need to know this in order to be able to calculate, e.g., how the orbit of a small black hole (~ 1 solar mass) inspirals into a a large black hole ($\sim 10^9$ solar mass). There is a general consensus that the gravitational self-force is given by the "MiSaTaQuWa equations": In the absence of "incoming radiation", the deviation from geodesic motion is given by

$$u^\nu \nabla_\nu u^\mu = -(g^{\mu\nu} + u^\mu u^\nu) u^\rho u^\lambda \left(\nabla_\rho h^{\rm tail}_{\lambda\nu} - \frac{1}{2} \nabla_\nu h^{\rm tail}_{\rho\lambda} \right)$$

where

$$h^{\rm tail}_{\mu\nu}(\tau) \equiv \int_{-\infty}^{\tau-} \left(G_{+\mu\nu\mu'\nu'} - \frac{1}{2} g_{\mu\nu} G_{+\ \ \alpha\mu'\nu'}^{\ \alpha} \right) u^{\mu'} u^{\nu'} d\tau' \ .$$

where G_+ is the retarded Green's function for the wave operator $\nabla^\alpha \nabla_\alpha \tilde{h}_{\mu\nu} - 2R^\alpha{}_{\mu\nu}{}^\beta \tilde{h}_{\alpha\beta}$. (Note that only the part of G_+ interior to the light cone contributes to $h^{\mathrm{tail}}_{\mu\nu}$.) However, all derivations contain some unsatisfactory features, which is hardly surprising in view of the fact that "point particles" do not make sense in general relativity.

- Derivations that treat the body as a point particle require unjustified "regularizations".
- Derivations using matched asymptotic expansions involve make a number of ad hoc and/or unjustified assumptions.
- The axioms of the Quinn-Wald axiomatic approach have not been shown to follow from Einstein's

equation.

- All of the above derivations employ at some stage a "phoney" version of the linearized Einstein equation with a point particle source, wherein the Lorenz gauge version of the linearized Einstein equation is written down, but the Lorenz gauge condition is not imposed.

How Should Gravitational Self-Force Be Derived?

A precise formula for gravitational self-force can hold only in a limit where the size, R, of the body goes to zero. Since "point-particles" (of non-zero mass) do not make sense in general relativity—collapse to a black hole would occur before a point-particle limit could be taken—the mass, M, of the body must also go to zero as $R \to 0$. In the limit as $R, M \to 0$, the worldtube of the body should approach a curve, γ, which should be a geodesic of the "background metric". The self-force should arise as the lowest order in M correction to γ.

This suggests that we consider a one-parameter family of solutions to Einstein's equation, $(g_{ab}(\lambda), T_{ab}(\lambda))$, with

$R(\lambda) \to 0$ and $M(\lambda) \to 0$ as $\lambda \to 0$.

But, what conditions should be imposed on $(g_{ab}(\lambda), T_{ab}(\lambda))$ to ensure that it corresponds to a body that is shrinking down to zero size, but is not undergoing wild oscillations, drastically changing its shape, or doing other crazy things as it does so?

Since we wish to consider "strong field objects" such as black holes, we shall impose our conditions on the "exterior field" of the body rather than the stress-enegy of the body itself.

Limits of Spacetimes

As a very simple, explicit example, consider a one-parameter family of Schwarzschild-deSitter metrics with $M = \lambda$

$$
\begin{aligned}
ds^2(\lambda) = & - & (1 - \frac{2\lambda}{r} - Cr^2)dt^2 \\
& + & (1 - \frac{2\lambda}{r} - Cr^2)^{-1}dr^2 + r^2 d\Omega^2
\end{aligned}
$$

If we take the limit as $\lambda \to 0$ at fixed coordinates (t, r, θ, ϕ) with $r > 0$, it is easily seen that we obtain the deSitter metric—with the deSitter spacetime worldline γ defined by $r = 0$ corresponding to the location of the black hole "before it disappeared".

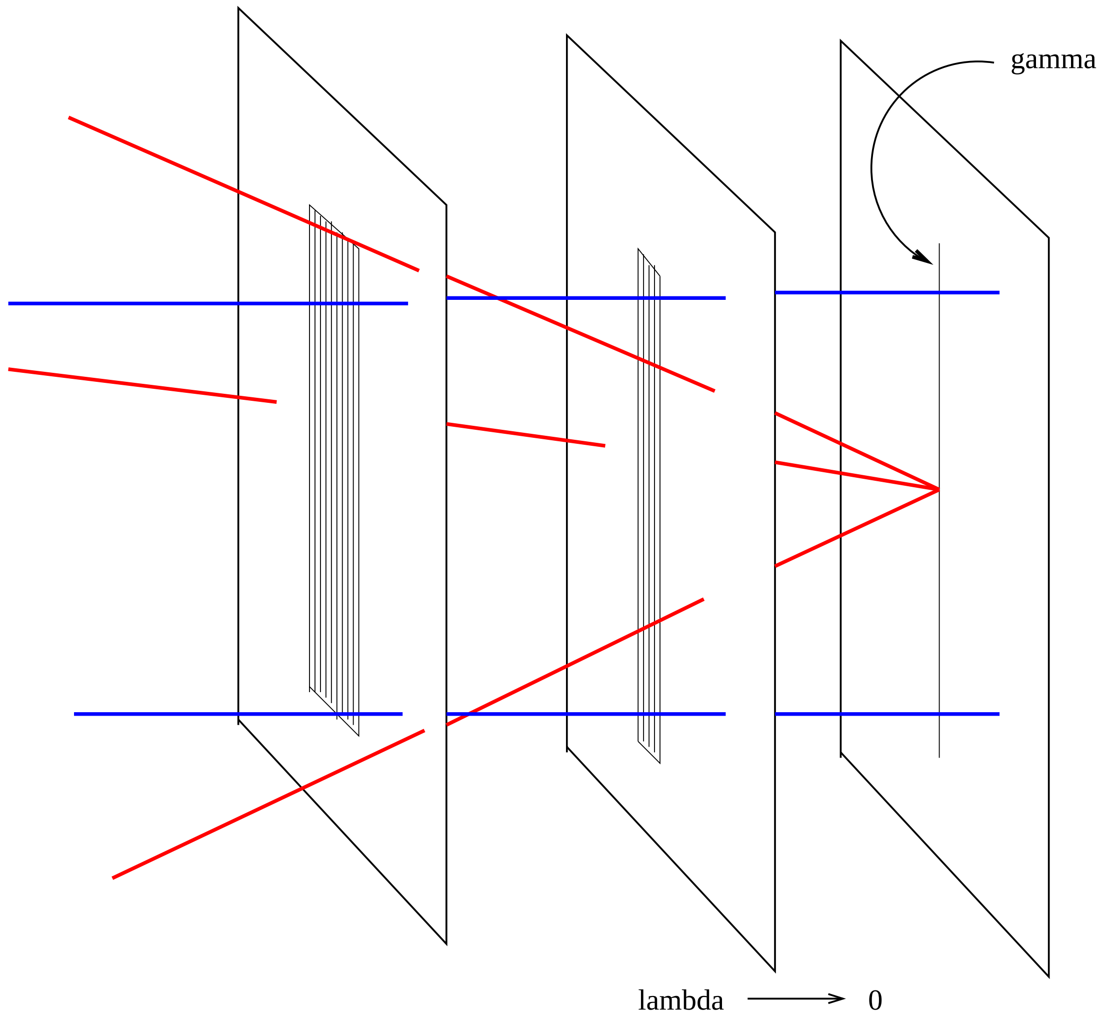
However, there is also another limit that can be taken. At each time t_0, can "blow up" the metric $g_{ab}(\lambda)$ by multiplying it by λ^{-2}, i.e., define $\bar{g}_{ab}(\lambda) = \lambda^{-2} g_{ab}(\lambda)$. Correspondingly rescale the coordinates by defining $\bar{r} = r/\lambda$, $\bar{t} = (t - t_0)/\lambda$. Then

$$\begin{aligned} d\bar{s}^2(\lambda) = & - & (1 - 2/\bar{r} - \lambda^2 C \bar{r}^2) d\bar{t}^2 \\ & + & (1 - 2/\bar{r} - \lambda^2 C \bar{r}^2)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \end{aligned}$$

In the limit as $\lambda \to 0$ (at fixed $(\bar{t}, \bar{r}, \theta, \phi)$) the "deSitter background" becomes irrelevant. The limiting metric is simply the Schwarzschild metric of unit mass. The fact that the limit as $\lambda \to 0$ exists can be attributed to the fact that the Schwarzschild black hole is shrinking to zero in a manner where, asymptotically, nothing changes except the overall scale.

The simultaneous existence of both types of limits contains a great deal of information about the one-parameter family of spacetimes $g_{\mu\nu}(\lambda)$.

Illustration of the Two Types of Limits



Our Basic Assumptions

We consider a one parameter family of solutions $g_{ab}(\lambda)$ satisfying the following properties:

- (i) Existence of the "ordinary limit": $g_{ab}(\lambda)$ is such that there exists coordinates x^α such that $g_{\mu\nu}(\lambda, x^\alpha)$ is jointly smooth in (λ, x^α), at least for $r > \bar{R}\lambda$ for some constant $\bar{R}$, where $r \equiv \sqrt{\sum(x^i)^2}$ $(i = 1, 2, 3)$. For all λ and for $r > \bar{R}\lambda$, $g_{ab}(\lambda)$ is a vacuum solution of Einstein's equation. Furthermore, $g_{\mu\nu}(\lambda = 0, x^\alpha)$ is smooth in x^α, including at $r = 0$, and, for $\lambda = 0$, the curve γ defined by $r = 0$ is timelike.
- (ii) Existence of the "scaled limit": For each t_0, we

define $\bar{t} \equiv (t - t_0)/\lambda$, $\bar{x}^i \equiv x^i/\lambda$. Then the metric $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{x}^\alpha) \equiv \lambda^{-2} g_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{x}^\alpha)$ is jointly smooth in $(\lambda, t_0; \bar{x}^\alpha)$ for $\bar{r} \equiv r/\lambda > \bar{R}$.

An Additional Uniformity Requirement

We have

$$\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{t}, \bar{x}^i) = g_{\mu\nu}(\lambda; t_0 + \lambda\bar{t}, \lambda\bar{x}^i) \, .$$

Introduce new the variable $\beta \equiv \lambda/r = 1/\bar{r}$. View each component of $g_{ab}(\lambda)$ in the coordinates x^α (or, equivalently, each component of $\bar{g}_{ab}(\lambda)$ in the coordinates $\bar{x}^\alpha$) as a function, f, of $(r, \beta, t, \theta, \phi)$, where θ and ϕ are defined in terms of x^i by the usual formula for spherical polar angles. Then, by assumption (ii) we see that for $0 < \beta < 1/\bar{R}$, f is smooth in $(r, \beta, t, \theta, \phi)$ (apart from the usual polar coordinate singularities at the poles) for all r *including* $r = 0$. By assumption (i), we see that for all

$r > 0$, f is smooth in $(r, \beta, t, \theta, \phi)$ for $\beta < 1/\bar{R}$, *including* $\beta = 0$. Furthermore, for $\beta = 0$, f is smooth in r, *including* $r = 0$.

We now impose the additional uniformity requirement on our one-parameter family of spacetimes: f is jointly smooth in $(r, \beta, t, \theta, \phi)$ at $r = \beta = 0$. We already know from our previous assumptions that $g_{\mu\nu}(\lambda; t, r, \theta, \phi)$ and its derivatives with respect to x^α approach a limit if we let $\lambda \to 0$ at fixed r and then let $r \to 0$. The uniformity requirement implies that the same limits are attained whenever λ and r both go to zero in any way such that λ/r goes to zero.

The uniformity requirement implies that in a neighborhood of $r = \beta = 0$ (with $r, \beta \geq 0$), we can uniformly approximate f by a series in r and β. This means that we can approximate $g_{\mu\nu}$ by a series in r and λ/r, i.e., we have

$$g_{\mu\nu}(\lambda; t, r, \theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} r^n \left(\frac{\lambda}{r}\right)^m (a_{\mu\nu})_{nm}(t, \theta, \phi)$$

This yields a far zone expansion. Equivalently, we have

$$\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{t}, \bar{r}, \theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} (\lambda \bar{r})^n \left(\frac{1}{\bar{r}}\right)^m (a_{\mu\nu})_{nm}(t_0 + \lambda \bar{t}, \theta, \phi)$$

Taylor expanding this formula with respect to the time variable yields a near zone expansion.

Since we can express $\bar{g}_{\bar{\mu}\bar{\nu}}$ at $\lambda = 0$ as a series in $1/\bar{r}$ as $\bar{r} \longrightarrow \infty$ and since $\bar{g}_{\bar{\mu}\bar{\nu}}$ at $\lambda = 0$ does not depend on $\bar{t}$, we see that $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$ is a stationary, asymptotically flat spacetime.

Geodesic Motion

The far zone expansion tells us that

$$g_{\mu\nu}(\lambda; t, r, \theta, \phi) = \sum_{n,m} (a_{\mu\nu})_{nm}(t, \theta, \phi) r^n \left(\frac{\lambda}{r}\right)^m$$

We choose the coordinates x^α so that at $\lambda = 0$ they correspond to Fermi normal coordinates about the worldline γ. It follows that near γ (i.e., for small r) the metric $g_{\mu\nu}$ must take the form

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r) + \lambda \left(\frac{C_{\mu\nu}(t, \theta, \phi)}{r} + O(1) \right) + O(\lambda^2)$$

Now, for $r > 0$, the coefficient of λ, namely

$$\gamma_{\mu\nu} = \frac{C_{\mu\nu}}{r} + O(1)$$

must satisfy the vacuum linearized Einstein equation off of the background spacetime $g_{\mu\nu}(\lambda = 0)$. However, since each component of $\gamma_{\mu\nu}$ is a locally L^1 function, it follows immediately that $\gamma_{\mu\nu}$ is well defined as a distribution. It is not difficult to show that, as a distribution, $\gamma_{\mu\nu}$ satisfies the linearized Einstein equation with source of the form $N_{\mu\nu}(t)\delta^{(3)}(x^i)$, where $N_{\mu\nu}$ is given by a formula involving the limit as $r \to 0$ of the angular average of $C_{\mu\nu}$ and its first derivative. The linearized Bianchi identity then immediately implies that (i) $N_{\mu\nu}$ is of the form $M(t)u_\mu u_\nu$, where u^μ is the unit tangent to γ; (ii) if $M \neq 0$, then γ is a geodesic; and (iii) M is independent of t.

Description of Motion to First Order

Since $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$ is an asymptotically flat spacetime, its "mass dipole moment" can be set to zero (at all t_0) as a gauge condition on the coordinates $\bar{x}^i$. The new coordinates $\bar{x}^i$ then have the interpretation of being "center of mass coordinates" for the spacetime $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$. In terms of our original coordinates x^α, the transformation to center of mass coordinates at all t_0 corresponds to a coordinate transformation $x^\alpha \to \hat{x}^\alpha$ of the form

$$\hat{x}^\alpha(t) = x^\alpha - \lambda A^\alpha(t, x^i) + O(\lambda^2)\,.$$

To first order in λ, the world line defined by $\hat{x}^i = 0$

should correspond to the “location” of the body. By solving $\hat{x}^i(t, Z^i(t)) = 0$, we see that to first order in λ the displacement, $Z^i(t)$, of the body from γ in the coordinates x^α is given by

$$Z^i(t) = \lambda A^i(t, x^j = 0) + O(\lambda^2)\,.$$

The quantity Z^i is most naturally interpreted as a “deviation vector field” defined on γ. Our goal is to derive relations (if any) that hold for Z^i that are independent of the choice of one-parameter family satisfying our assumptions.

Form of the Metric to Order λ^2

We choose the x^α coordinates to correspond to the harmonic gauge to first order in λ. To order λ^2, the leading order in r terms in $g_{\alpha\beta}$ are,

$$\begin{aligned}
& g_{\alpha\beta}(\lambda; t, x^i) = \eta_{\alpha\beta} + B_{\alpha i\beta j}(t)x^i x^j + O(r^3) \\
+ \quad & \lambda\left(\frac{2M}{r}\delta_{\alpha\beta} + h^{\rm tail}_{\alpha\beta}(t) + h^{\rm tail}_{\alpha\beta,i}(t)x^i + M\mathcal{R}_{\alpha\beta}(t) + O(r^2)\right) \\
+ \quad & \lambda^2\left(\frac{M^2}{r^2}\left(-2t_\alpha t_\beta + 3n_\alpha n_\beta\right) + \frac{2}{r^2}P_i(t)n^i\delta_{\alpha\beta}\right. \\
+ \quad & \left.\frac{1}{r^2}t_{(\alpha}S_{\beta)j}(t)n^j + \frac{1}{r}K_{\alpha\beta}(t,\theta,\phi) + H_{\alpha\beta}(t,\theta,\phi) + O(r)\right) \\
& +O(\lambda^3)
\end{aligned}$$

Here B and $\mathcal{R}$ are expressions involving the curvature of $g_{\mu\nu}(\lambda = 0)$ and we have introduced the "unknown" tensors K and H. For simplicity, we have assumed no "incoming radiation". Hadamard expansion techniques and 2nd order perturbation theory were used to derive this expression.

Form of the Scaled Metric to Order λ^2

Using the coordinate shift $x^\mu \to x^\mu - A^\mu$ to cancel the mass dipole term, the above expression translates into the following expression for the scaled metric

$$\begin{aligned}\bar{g}_{\bar{\alpha}\bar{\beta}}(\hat{t}_0) = \eta_{\alpha\beta} \quad &+ \quad \frac{2M}{\bar{r}}\delta_{\alpha\beta} + \frac{M^2}{\bar{r}^2}\left(-2t_\alpha t_\beta + 3n_\alpha n_\beta\right) \\ &+ \quad \frac{1}{\bar{r}^2} t_{(\alpha} S_{\beta)j} n^j + O\left(\frac{1}{\bar{r}^3}\right)\end{aligned}$$

$$\begin{aligned}&+ \quad \lambda[h^{\mathrm{tail}}_{\alpha\beta} + 2A_{(\alpha,\beta)} + \frac{1}{r}K_{\alpha\beta} \\ &+ \quad \frac{\bar{t}}{\bar{r}^2} t_{(\alpha}\dot{S}_{\beta)j} n^j + O\left(\frac{1}{\bar{r}^2}\right) + \bar{t}\; O\left(\frac{1}{\bar{r}^3}\right)\end{aligned}$$

$$
\begin{aligned}
&+ \quad \lambda^2[B_{\alpha i\beta j}\bar{x}^i\bar{x}^j + h^{\mathrm{tail}}_{\alpha\beta,\gamma}\bar{x}^\gamma + M\mathcal{R}_{\alpha\beta}(\bar{x}^i) + 2B_{\alpha i\beta j}A^i\bar{x}^j \\
&+ \quad 2A_{(\alpha,\beta)\gamma}\bar{x}^\gamma + H_{\alpha\beta} + \frac{\bar{t}}{\bar{r}}\dot{K}_{\alpha\beta} + \frac{\bar{t}^2}{\bar{r}^2}t_{(\alpha}\ddot{S}_{\beta)j}n^j \\
&+ \quad O\left(\frac{1}{\bar{r}}\right) + \bar{t}\; O\left(\frac{1}{\bar{r}^3}\right) + \bar{t}^2\; O\left(\frac{1}{\bar{r}^3}\right)]
\end{aligned}
$$

$$
+O(\lambda^3)\ .
$$

Results

The first order (black) terms satisfy the linearized vacuum Einstein equation about the background (blue) "near zone" metric. From this equation, we find that $dS_{ij}/dt = 0$, i.e., to lowest order, spin is parallelly propagated along γ.

The second order (red) terms satisfy the linearized Einstein equation about the background (blue) "near zone" metric with source given by the second order Einstein tensor of the first order (black) terms. Extracting the $\ell = 1$, electric parity, even-under-time-reversal part of this equation that is leading order in $\bar{r}$, we obtain (after considerable algebra!)

$$A_{i,00} = \frac{1}{2M} S^{kl} R_{kl0i} - R_{0j0i} A^j - \left(h^{\mathrm{tail}}_{i0,0} - \frac{1}{2} h^{\mathrm{tail}}_{00,i} \right) \ .$$

In other words, in the Lorenz gauge, the deviation vector field, Z^a, on γ that describes the first order perturbation to the motion satisfies

$$\begin{aligned} u^c \nabla_c (u^b \nabla_b Z^a) &= \frac{1}{2M} R_{bcd}{}^a S^{bc} u^d - R_{bcd}{}^a u^b u^d Z^c \\ &- (g^{ab} + u^a u^b)(\nabla_d h^{\mathrm{tail}}_{bc} - \frac{1}{2} \nabla_b h^{\mathrm{tail}}_{cd}) u^c u^d \ . \end{aligned}$$

Beyond Perturbation Theory

We have now obtained the perturbative correction to geodesic motion due to spin and self-force effects. However, at late times, the orbit will deviate significantly from the unperturbed geodesic, so our perturbative description will not be accurate. Clearly, going to any (finite) higher order in perturbation theory will not help (much).

However, if the mass and size of the body are sufficiently small, we expect that its motion is well described *locally* as a small perturbation of *some* geodesic. Therefore, one should obtain a good description of the motion by making up (!!) a "self-consistent perturbative equation" that satisfies the following criteria: (1) It has a well posed intial value formulation. (2) It has the same "number of degrees of freedom" as the original system. (3) It's solutions correspond closely to the solutions of the of the original perturbation equation over a time interval where the perturbation remains small. In some sense, such a self-consistent perturbative equation would take into account the important ("secular") higher order perturbative effects (to all orders), but ignore other higher order corrections. The MiSaTaQuWa equations are a good candidate for such a self-consistent perturbative equation.

Summary and Conclusions

- Our only assumptions were the existence of a one-parameter family satisfying assumptions (i) and (ii) and the "uniformity requirement".
- We showed that at lowest ("zeroth") order, the motion of a "small" body is described by a geodesic, γ, of the "background" spaceetime. We then derived is a formula for the first order deviation of the "center of mass" worldline of the body from γ.
- The MiSaTaQuWa equations (together with "spin force" terms") arise as (candidate) "self-consistent perturbative equations" based on our first order

perturbative result. It is only at this stage that "phoney" Einstein equations come into play.

- It should be possible to use this formalism to take higher order corrections to the motion into account in a systematic way.