Inequalities for the ADM-mass and capacity of asymptotically flat manifolds with minimal boundary

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Abstract. We present some recent developments involving inequalities for the ADM-mass and capacity of asymptotically flat manifolds with boundary. New, more general proofs of classic Euclidean estimates are also included. The inequalities are rigid and valid in all dimensions, and constitute a step towards proving the Riemannian Penrose inequality in arbitrary dimensions.

1. Introduction

An *asymptotically flat* manifold is a Riemannian manifold which, outside a compact set, is diffeomorphic to the complement of a ball in Euclidean space. Furthermore, in the asymptotic coordinates induced by the diffeomorphism, the metric and its derivatives decay fast enough to the flat metric of Euclidean space.

Asymptotically flat manifolds play an important role in relativity, as they are the best model for spacelike slices of *isolated* gravitational systems. Much work has been done on these types of manifolds, particularly under the extra hypothesis that they have *nonnegative scalar curvature.* This requirement corresponds, in physical terms, to the simpler situation of being a spacelike slice inside a *time-symmetric* asymptotically flat spacetime satisfying the dominant energy condition.

Several questions remain open regarding such asymptotically flat manifolds, especially in high dimensions. Among the two most prominent ones are: (a) the *positive mass theorem*, and (b) the *Penrose inequality*, neither of which is known to hold, in full generality, *in dimensions eight or above.* Furthermore, these are among the most important unanswered questions in all of geometric analysis because of their implications to related problems.

A special case of asymptotically flat manifolds are those that are also *conformally flat.* These correspond to manifolds globally conformal to Euclidean space removed a region (possibly empty). Question (a) from above –positive mass theorem– is known to hold in all dimensions for these manifolds. The proof of this uses the rather straightforward argument. (See §2). By analogy to this case, Bray and Iga conjectured in [3] that the answer to (b) –Penrose inequality– should also be relatively straightforward to prove, in all dimensions, under these hypotheses. Nevertheless, no such proof is known to exist. Partial progress towards proving the Penrose inequality for conformally flat manifolds has occurred recently. Our goal

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here is to describe these developments in a unified way, sketching some of the proofs in the literature.

2. Basic setup and motivation

We first introduce some general notions. In what follows (M^n, g) denotes a Riemannian manifold of dimension $n \geq 3$.

Definition. Suppose (M^n, g) as above is a complete, non-compact (with or without boundary) manifold, and has only one end $\mathcal{E}$. (This assumption is only for the sake of simplicity.) We say that (M, g) is *asymptotically flat* if outside a compact set, it is diffeomorphic to the complement of a ball in Euclidean space, and in the coordinates given by this diffeomorphism, the metric satisfies the asymptotic conditions

$$|g - \delta| = O(|x|^{-p}), \quad |\partial g| = O(|x|^{-p-1}), \quad |\partial^2 g| = O(|x|^{-p-2}).$$

Here, $\delta = \delta_{ij}$ is the flat metric of Euclidean space, and $p > \frac{n-2}{2}$.

A notion of total mass, called the ADM-mass, can be defined for asymptotically flat manifolds as follows.

Definition (ADM-mass). Let (M, g) be an asymptotically flat manifold. Its ADM-mass is

$$m = m_{ADM}(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj}) \nu^j \, d\sigma_r^0.$$

Here, S_r is a Euclidean coordinate sphere, and $d\sigma_r^0$ is Euclidean surface area.

It is well known that under the above asymptotic conditions, the ADM-mass is well-defined independently of the asymptotically flat coordinates (cf. [1, 8]).

Definition (CF-manifold). A manifold (M, g) is said to be conformally flat if it is isometric to $(\mathbb{R}^n \setminus \Omega, u^{4/(n-2)}\delta_{ij})$, where $u > 0$ is a smooth function defined outside the smooth (possibly empty) region $\Omega \subset \mathbb{R}^n$. We say that a manifold (M, g) is a *CF-manifold* if it is asymptotically flat, conformally flat, *has nonnegative scalar curvature*, u is harmonic outside a compact subset, and $u \to 1$ as $|x| \to \infty$, where $g = u^{4/(n-2)}\delta_{ij}$.

The normalization $u \to 1$ at infinity is for simplicity, as we see in the next section. The condition that u be harmonic at infinity simplifies the calculations quite a bit, but is not necessary.

Lemma 1 (Schoen-Yau [19]). *Let (M, g) be an AF, conformally flat manifold with nonnegative scalar curvature. Then given any $\epsilon > 0$, there exists a CF-metric g_0 with nonnegative scalar curvature such that*

$$1 - \epsilon \leq \frac{g_0(v, v)}{g(v, v)} \leq 1 + \epsilon,$$

for all nonzero vectors v in the tangent space at every point in M, and so that

$$|m - m_0| \leq \epsilon,$$

where m and m_0 are the ADM masses of (M, g) and (M, g_0), respectively.

The reason for requiring that u be harmonic outside a compact set is the following well known lemma.

LEMMA 2. *Let (M, g) be a CF-manifold as above. Then u admits the expansion*

$$u(x) = 1 + \frac{m}{2}|x|^{2-n} + l.o.t,$$

where m is the ADM-mass of M.

The proof of this lemma is an easy exercise using spherical harmonics. (See e.g. §2 of [2].) Notice that because of it, it is desirable to work with metrics that are harmonically flat near infinity to simplify calculations involving the ADM-mass. The following lemma follows directly from a result of Schoen and Yau that gives that AF-manifolds may be approximated by harmonically flat AF-maniolds.

EXAMPLE. The prototypical example of a CF-manifold, aside from Euclidean space itself, is the so-called *Riemannian Schwarzschild manifold* of mass $m > 0$. It is defined as the manifold $(\mathbb{R}^n \setminus B, u^{4/(n-2)}\delta_{ij})$, where B is the Euclidean ball of radius $R = (m/2)^{1/(n-2)}$ (called Schwarzschild radius), and

$$u = u(|x|) = 1 + \frac{m}{2}|x|^{2-n}.$$

The following properties of the Riemannian Schwarzschild manifold of mass $m > 0$ are well known:

- (i) it is asymptotically flat, and scalar flat;
- (ii) its ADM-mass is $m > 0$, as the name suggests;
- (iii) the boundary ∂B is minimal;
- (iii)' (Actually, ∂B is totally geodesic since it is fixed by the isometry of $(\mathbb{R}^n \setminus \{0\}, g)$ given by $x \mapsto x/|x|^2$. Here, g is the above metric extended to $\mathbb{R}^n$ minus the origin using the same formula);
- (iv) the boundary ∂B is outer area minimizing, and there are no other minimal hypersurfaces in it;
- (v) the area of the boundary ∂B (with respect to the metric g) is exactly $A = \omega_{n-1}(2m)^{(n-1)/(n-2)}$. Equivalently, it satisfies the equality case of the *Riemannian Penrose* inequality

$$m \geq \frac{1}{2}(A/\omega_{n-1})^{(n-2)/(n-1)},$$

where ω_{n-1} is the area of the $(n-1)$-dimensional unit round sphere.

We are now ready to prove the theorem which motivates the study of CF-manifolds.

THEOREM 3 (Positive Mass Theorem for CF-manifolds). *Let (M, g) be a CF-manifold without boundary. Then $m \geq 0$, with equality if and only if the manifold is Euclidean space.*

PROOF. Using the above lemma it follows that it suffices to prove the PMT for metrics that are harmonically flat near infinity. Let $g = u^{4/(n-2)}\delta_{ij}$. Since (M, g) has nonnegative scalar curvature we know that $\Delta u \leq 0$. Integrating $-u\Delta u$ over M and applying the divergence theorem gives that $m \geq 0$. Equality $m = 0$ implies $u \equiv 1$. □

Remark 4. The general version of the positive mass theorem (PMT) is the same as the one above but with 'CF-manifold' replaced by 'AF-manifold with nonnegative scalar curvature.' The known proof of the (general) positive mass theorem works for manifolds of dimension less than eight. It was done by Schoen and Yau in [18] using geometric measure theory and minimal surface techniques. The (general) PMT was extended to arbitrary-dimensional asymptotically flat Riemannian manifolds which are also *spin* by Witten in [22]; and to asymptotically flat graphs over $\mathbb{R}^n$ inside $\mathbb{R}^{n+1}$ by Lam [16].

The positive mass theorem has many beautiful applications, as well as a clear physical meaning. Indeed, it allows us to interpret the ADM mass of an asymptotically flat manifold as its total mass, since the theorem argues that a slice of spacetime with nonnegative energy density has nonnegative total mass. Also, from a Riemannian-geometry perspective, the theorem is a *rigid inequality* which provides with a characterization of Euclidean space as the equality case of the inequality $m \geq 0$. The positive mass theorem in high dimensions remains one of the main open problems in geometric analysis.

3. Inequalities for the ADM-mass and capacity of CF-manifolds with minimal boundary

One of the main open problems in geometric analysis is the proof of the Riemannian Penrose inequality in high dimensions. This inequality can be thought of as a refinement of the PMT for manifolds whose boundary is an outermost minimal hypersurface. The general conjecture is the following.

Conjecture (Riemannian Penrose Inequality). *Let (M, g) be an AF-manifold with boundary having nonnegative scalar curvature. Let m denote its mass. Assume that its boundary is an outermost minimal hypersurface of area A. Then*

$$m \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{RPI}$$

with equality if and only if (M, g) is the Riemannian Schwarzschild manifold (see the Example in the previous section).

The RPI was first proved in three-dimensions by Huisken and Ilmanen [13] with the limitation that the term A above refers to the area of any one *component* of the boundary, rather than the area of the boundary as a whole. This was done using Geroch's monotonicity formula for the Hawking mass under inverse mean curvature flow of [10]. Geroch's formula relies heavily on Gauss-Bonnet so it does not generalize to higher dimensions.

A different approach to proving the RPI was given by Bray in [2]. He was able to prove the *full* RPI in dimension three (i.e. the case of A above denoting the area of the whole boundary) using a conformal flow of the metric. Bray's argument was extended to dimensions less than eight by Bray-Lee in [5] (with rigidity only for spin manifolds). Bray-Lee's argument cannot be directly extended to dimensions eight and above since: (i) it uses techniques from geometric measure theory, and it is known that singularities appear inside the area-minimizing hypersurfaces used in the proof; and (ii) it uses the PMT, which remains unproved (in full generality) in high dimensions.

In view of the relatively straightforward proof of the PMT in the CF-case, Bray and Iga conjectured in [3] that the RPI should also hold for CF-manifolds. Furthermore, they assert that one should be able to prove this version of the RPI, in all dimensions, using classical techniques alone. In that work, the following theorem is proved using classical PDE techniques *for dimension three.*

THEOREM 5 (Bray-Iga [3]). *Suppose (M^3, g) is a CF-manifold where $M = \mathbb{R}^3 \setminus \{0\}$ and $g = u^4\delta_{ij}$. Assume u has a pole at $x = 0$, and that every smooth surface $\Sigma \subset \mathbb{R}^3$ which bounds an open set containing the origin has g-area greater than or equal to A. Then*

$$m \geq \lambda\sqrt{A},$$

for some universal constant $\lambda > 0$.

An important tool developed by Bray for proving the RPI is the so-called mass-capacity inequality. It uses the notion of capacity of the boundary of an AF-manifold, which is defined as follows.

DEFINITION (Capacity). Let (M^n, g) be an AF manifold with boundary Σ. The capacity of Σ is the quantity

$$\mathrm{Cap}_g(\Sigma) = \inf_{\varphi \in M_0^1} \left\{ \frac{1}{(n-2)\omega_{n-1}} \int_M |\nabla_g \varphi|_g^2 dV_g \right\},$$

where the infimum is taken over the set M_0^1 of all smooth functions on M which are exactly 0 on Σ and approach 1 at infinity in the AF end of M.

REMARK 6. Changing the boundary conditions we could also define (for $a \neq b$):

$$Cap_g^{(a,b)}(\Sigma) = \inf_{\varphi \in M_a^b} \left\{ \frac{1}{(n-2)\omega_{n-1}} \int_M |\nabla_g \psi|_g^2 dV_g \right\},$$

where the infimum is now taken over the set M_a^b of all smooth functions on M which are exactly a on Σ and approach b at infinity in the AF end of M. Since the map $\psi \mapsto \frac{a-\psi}{a-b}$ defines a bijection between $M_a^b \to M_0^1$ which scales the integral of the square of the gradient by a constant, it follows that $Cap_g^{(a,b)}(\Sigma) = (a-b)^2 Cap_g(\Sigma)$. (We will use this fact in the proof of part (IV) of Theorem 12 below.)

The mass-capacity inequality states the following.

THEOREM 7 (Bray [2]). *Let (M^3, g) be an AF manifold with nonnegative scalar curvature and minimal boundary Σ. Then the ADM mass satisfies*

$$m \geq \mathrm{Cap}_g(\Sigma),$$

with equality if and only if the manifold is isometric to the Riemannian Schwarzschild manifold of mass m.

The mass-capacity inequality was proved by Bray in [2] using the PMT and the reflection argument of [7]. A proof that is valid for CF-manifolds in all dimension was given by the author in [20]. The argument is based on extending Bray's original proof. In order to do that one first shows that CF-manifolds are spin, and then adapts Bray's proof using Witten's version of the PMT for spin manifolds.

It is worth pointing out that related theorem to the mass-capacity inequality of above was proved by Bray and Miao in [6]. There, the authors find an upper bound for the capacity of the boundary of a three-dimensional AF-manifold with nonnegative scalar curvature in terms of a function of the Hawking mass of the boundary.

3.1. Volumetric Penrose inequality. Bray's proof of the mass-capacity inequality uses the positive mass theorem. Actually, the only place in Bray's work of [2] where the PMT is used is to prove this inequality. Recently, Freire and the author gave a proof of this inequality for the CF-case *which does not rely on the PMT.* (See (a) of Theorem 11 below.) We present an application of the mass-capacity inequality to prove the "volumetric" Penrose inequality.

THEOREM 8 (Schwartz [20]). *Let (M^n, g), be an CF-manifold with g-minimal boundary $\Sigma = \partial\Omega$. Assume further that Σ is mean-convex with respect to the Euclidean metric. Then the ADM mass of M satisfies*

$$m \geq \left(\frac{V_0}{\beta_n}\right)^{\frac{n-2}{n}},$$

where V_0 is the Euclidean volume of Ω and β_n is the volume of the round unit ball in $\mathbb{R}^n$.

In order to prove the theorem we first prove the following useful fact.

LEMMA 9 (Schwartz [20]). *Let (M, g) as in Theorem 8, with $g = u^{4/(n-2)}\delta_{ij}$. Then $u \geq 1$.*

PROOF. Recall that the transformation law for the scalar curvature under conformal changes of the metric is given by $R_g = \frac{4(n-1)}{n-2}u^{-(n+2)/(n-2)}(-\Delta_0 + \frac{n-2}{4(n-1)}R_0)u$, where Δ_0 is the Euclidean Laplacian and R_0 is the Euclidean scalar curvature, namely $R_0 \equiv 0$. Since we assume that $R_g \geq 0$, it follows that u is superharmonic on M. Therefore, u achieves its minimum value at either infinity or at the boundary $\partial\Omega$. At infinity u goes to one. We now show that at the boundary it does not achieve its minimum, and so it must be everywhere greater or equal than one.

Claim. u does not achieve its minimum on the boundary $\partial\Omega$.

From hypothesis, the boundary of M is a minimal hypersurface. This is, the mean curvature of the boundary of M is zero with respect to the metric $g = u^{4/(n-2)}\delta_{ij}$. Now, the transformation law for the mean curvature under the conformal change of the metric $g = u^{4/(n-2)}\delta_{ij}$ is given by $h_g = \frac{2}{n-2}u^{-n/(n-2)}(\partial_\nu + \frac{(n-2)}{2}h_0)u$, where h_0 is the Euclidean mean curvature and ν is the outward-pointing normal. Since we have assumed that the boundary of Ω is mean convex, i.e. that $h_0 > 0$, it follows that $\partial_\nu u < 0$ on all of the boundary of Ω. This way, u decreases when we move away from the boundary towards the interior of M. From this it follows that u cannot achieve its minimum on the boundary. This proves the claim, and the Lemma follows. □

Another ingredient we need to prove Theorem 8 is a classical fact about spherical rearrangements by Pólya and Szegö [17]. (See also [21], [12].) The idea is the following. Let u be a function in $W^{1,p}(\mathbb{R}^n)$. Its *spherical decreasing rearrangement*, $u^*(x) \equiv u^*(|x|)$, is the unique radially symmetric function on $\mathbb{R}^n$ which is decreasing on $|x|$, and so that the Lebesgue measure of the super-level sets of u^* equals the Lebesgue measure of the super-level sets of u. More precisely, u^* is defined as the unique decreasing spherically symmetric function on $\mathbb{R}^n$ so that $\mu\{u \geq K\} = \mu\{u^* \geq K\}$ for all $K \in \mathbb{R}$.

Theorem 10 (Pólya-Szegö [17]). *Spherical decreasing rearrangement preserves L^p norms and decreases $W^{1,p}$ norms.*

Proof of Theorem 8. Bray's mass-capacity theorem (and its extension by the author in [20]) gives $m \geq Cap_g(\Sigma)$. To estimate the capacity we use the following argument. We first note that, without loss of generality, we may assume that the functions φ in the definition of the capacity satisfy $0 \leq \varphi \leq 1$. Consider any such function and extend it to a function $\tilde{\varphi}$ defined on all of $\mathbb{R}^n$ by setting it to be exactly zero inside Ω. The resulting function $\tilde{\varphi}$ is Lipschitz and satisfies $\int_{\mathbb{R}^n \setminus \Omega} |\nabla\varphi|^2 dV = \int_{\mathbb{R}^n} |\nabla\tilde{\varphi}|^2 dV$, where all the integrands are with respect to the Euclidean metric. Consider the spherical decreasing rearrangement of $\tilde{\varphi}$ given by $(\tilde{\varphi})^*$. By the Pólya-Szegö theorem we have that $\int_{\mathbb{R}^n} |\nabla\tilde{\varphi}|^2 dV \geq \int_{\mathbb{R}^n} |\nabla(\tilde{\varphi})^*|^2 dV$. We deduce

$$\int_{\mathbb{R}^n \setminus \Omega} |\nabla\varphi|^2 dV \geq \int_{\mathbb{R}^n} |\nabla(\tilde{\varphi})^*|^2 dV. \tag{3.1}$$

A standard calculation shows that whenever $u \geq 1$ we get $\int_M |\nabla_g\varphi|_g^2 dV_g \geq \int_M |\nabla\varphi|^2 dV$. On the other hand, notice that $(\tilde{\varphi})^*$ is exactly zero on the ball B_R, where $R = (V_0/\beta_n)^{1/n}$ is the radius of the ball of volume $V_0 = |\Omega|$. It follows by the definition of Euclidean capacity that

$$\int_{\mathbb{R}^n} |\nabla(\tilde{\varphi})^*|^2 dV = \int_{\mathbb{R}^n \setminus B_R} |\nabla(\tilde{\varphi})^*|^2 dV \geq Cap_{Eucl}(\Sigma).$$

These last two inequalities together with equation (3.1) give that $\int_M |\nabla_g\varphi|_g^2 dV_g \geq \int_{\mathbb{R}^n \setminus B_R} |\nabla(\tilde{\varphi})^*|^2 dV$. Taking infimum over φ one obtains $Cap_g(\Sigma) \geq Cap_{Eucl}(B_R)$. An easy calculation gives that this last quantity is exactly $(V_0/\beta_n)^{(n-2)/n}$. This ends the proof. □

3.2. Sharp mass-capacity and volumetric Penrose inequalities. Theorem 8 has two limitations: on the one hand, the volumetric Penrose inequality is not sharp nor does it contain a rigidity statement; on the other hand, the lower bound for the ADM mass it gives is in terms of an Euclidean volume.

In a recent joint work with Freire [9], the author was able to sharpen Theorem 8 to a rigid inequality which also contains a new proof of the mass-capacity inequality for conformally-flat manifolds that *does not use the positive mass theorem.* The precise statement is the following.

Theorem 11 (Freire-Schwartz [9]). *Let (M, g) be a CF-manifold as above with $u_{\partial M} \geq 2$, and let m denote its ADM mass.*

(a) **Mass-capacity inequality:** $m \geq \mathrm{Cap}_g(\Sigma)$. *Equality holds if and only if g is the Riemannian Schwarzschild metric.*
(b) **Sharp Volumetric Penrose inequality:** $m \geq 2\,(V_0/\beta_n)^{(n-2)/n}$, *where V_0 is the Euclidean volume of Ω, and β_n is the volume of the Euclidean unit n-ball. Equality holds if and only if g is the Riemannian Schwarzschild metric.*

The requirement $u_{\partial M} \geq 2$ in the above Theorem is for technical reasons. Actually, the Theorem also holds whenever u is less than 2 on the boundary provided it does not oscillate much. We refer the reader to the paper [9] for more details. The proof Theorem 11 relies only on classical variational and PDE methods, as well as on Huisken-Ilmanen's high-dimensional inverse mean curvature flow [14]. This is the only application of Huisken-Ilmanen's high-dimensional IMCF to date. The theorem follows from the following technical result, as we see below.

THEOREM 12 (Freire-Schwartz [9]). *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain with boundary $\Sigma = \partial\Omega$, not necessarily connected. Let (M, g) be isometric to a conformally flat metric $g_{ij} = u^{\frac{4}{n-2}}\delta_{ij}$ on Ω^c which is asymptotically flat with ADM mass m. (Here $u > 0$ and $u \to 1$ towards infinity.) Assume further that (M, g) has non-negative scalar curvature $R_g \geq 0$. Then*

(I) If Σ is Euclidean mean-convex ($H_0 > 0$) and g-minimal ($H_g = 0$), then

$$C_0(\Sigma) < C_g(\Sigma) \leq C_0(\Sigma) + \frac{m}{2}.$$

Equality occurs in the second inequality if and only if u is harmonic.

(II) (Euclidean estimate.) Assume $H_0 > 0$ on Σ. Then:

$$C_0(\Sigma) \leq \frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0.$$

Equality holds if and only if Σ is a round sphere.

(III) Let $\alpha = \min_\Sigma u$. Under the same assumptions on Σ as in (I), we have:

$$\frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0 \leq \frac{m}{\alpha}.$$

Equality holds if and only if u is harmonic and constant on Σ (and, in this case, $\alpha \geq 2$.)(Note that by Lemma 9, $\alpha > 1$ always.)

(IV) Under the same assumptions on Σ as in (I), assume further $\alpha \geq 2$. Then

$$C_0(\Sigma) \leq \frac{m}{2}.$$

(V) (Euclidean estimate.) Assume $H_0 > 0$ on Σ, and Σ is outer-minimizing in $\mathbb{R}^n$ with area A. Then:

$$\frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0 \geq \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Equality holds if and only if Σ is a round sphere.

We prove only parts (II), (III) and (V) here. Part (IV) follows directly from (II) and (III), but for the case of $\alpha < 2$ with small oscillation more work is required. The full proof of the above theorem in its most general form can be found in [9].

PROOF OF THEOREM 12 PART (II). We use a modification of the method of Bray-Miao described in [6]. First, we get an upper bound for $\mathrm{Cap}_0(\Sigma)$ using test functions of the form $\varphi = f \circ \phi$, where $\phi \in C^1(\Omega^c, \mathbb{R}_+)$ is a (soon to be determined) proper function vanishing on $\Sigma = \Sigma_0$ whose level sets define a foliation $(\Sigma_t)_{t\geq 0}$ of

Ω^c, and $f : \mathbb{R} \to \mathbb{R}$ is any function that satisfies $f(0) = 0, f(\infty) = 1$. As noted in [6], we have

$$(n-2)\omega_{n-1}\mathrm{Cap}_0(\Sigma) \leq \inf \left\{ \int_0^\infty (f')^2 w(t) dt \ : \ f(0) = 0, f(\infty) = 1 \right\}, \tag{3.2}$$

where $w(t) := \int_{\Sigma_t} |\nabla_0 \phi| d\sigma_t^0 > 0$.

For the sake of simplicity let us omit the subscript/superscript '0' for the remainder of the proof. Moving away from the method of [6], we note that the one-dimensional variational problem (3.2) is easily solved.

Claim. Provided $w^{-1} \in L^1(0,\infty)$, the infimum of the right hand side of (3.2) equals $\mathcal{I}^{-1} = (\int_0^\infty \frac{1}{w(s)} ds)^{-1}$, and is attained by the function $f(t) = \frac{1}{\mathcal{I}} \int_0^t w^{-1}(s) ds$.

Proof. This follows from

$$1 = \int_0^\infty f' dt = \int_0^\infty f' w^{1/2} w^{-1/2} dt \leq \left(\int_0^\infty (f')^2 w(t) dt \right)^{1/2} \left(\int_0^\infty w^{-1}(t) dt \right)^{1/2}.$$

□

Consider the foliation $(\Sigma_t)_{t\geq 0}$ defined by the level sets of the function given by Huisken and Ilmanen's inverse mean curvature flow [13, 14] in $\Omega^c \subset \mathbb{R}^n$. We recall the summary given in [6] (which holds in all dimensions):

Theorem 13 (Huisken-Ilmanen, [13, 14]).

- *There exists a proper, locally Lipschitz function $\phi \geq 0$ on Ω^c, $\phi_{|\Sigma} = 0$. For $t > 0$, $\Sigma_t = \partial\{\phi \geq t\}$ and $\Sigma_t' = \partial\{\phi > t\}$ define increasing families of $C^{1,\alpha}$ hypersurfaces;*
- *The hypersurfaces Σ_t (resp.Σ_t') minimize (resp. strictly minimize) area among surfaces homologous to Σ_t in $\{\phi \geq t\} \subset \Omega^c$. The hypersurface $\Sigma' = \partial\{\phi > 0\}$ strictly minimizes area among hypersurfaces homologous to Σ in Ω^c.*
- *For almost all $t > 0$, the weak mean curvature of Σ_t is defined and equals $|\nabla\phi|$, which is positive a.e. on Σ_t.*

From Theorem 13 and the Claim from above it follows that

$$(n-2)\omega_{n-1} C_0(\Sigma) \leq \left(\int_0^\infty w^{-1}(t) dt \right)^{-1}, \text{ where } w(t) := \int_{\Sigma_t} H d\sigma_t. \tag{3.3}$$

Lemma 14 ([9]). *Consider the foliation $\{\Sigma_t\}$ given by IMCF in $\Omega^c \subset \mathbb{R}^n$ as above. Then*

$$\int_{\Sigma_t} H d\sigma \leq \left(\int_{\Sigma_0} H d\sigma \right) e^{\frac{n-2}{n-1} \cdot t}, \text{ for } t \geq 0.$$

Remark 15. Note that equality holds in the above inequality for the foliation by IMCF outside a sphere, which is given by $\Sigma_t = \partial B_{R(t)} \subset \mathbb{R}^n$, where $R(t) = e^{\frac{t}{n-1}}$.

Proof of Lemma 14. From [13] we have that, so long as the evolution remains smooth,

$$\frac{d}{dt} \left(\int_{\Sigma_t} H d\sigma_t \right) = \int_{\Sigma_t} \left(H - \frac{|A|^2}{H} \right) d\sigma_t \leq \frac{n-2}{n-1} \int_{\Sigma_t} H d\sigma_t, \tag{3.4}$$

where A denotes the second fundamental form, and the second inequality follows from

$$H - \frac{|A|^2}{H} - \frac{n-2}{n-1}H = \frac{1}{(n-1)H}(H^2 - (n-1)|A|^2) \leq 0. \tag{3.5}$$

(Note that equality occurs in this last inequality if and only if each connected component of Σ_t is a sphere.) It is easy to see that the inequalities extend through countably many jump times since the total mean curvature does not increase at the jump times. □

By straightforward integration, Lemma 14 implies:

$$\left(\int_0^\infty w^{-1}(t)dt\right)^{-1} \leq \frac{n-2}{n-1}\int_\Sigma H d\sigma.$$

Together with equation (3.3) this gives $(n-1)\omega_{n-1}C_0(\Sigma) \leq \int_\Sigma H d\sigma$, as claimed in part (II) of the Theorem.

Rigidity. From Remark 15 it follows that the inequality of part (II) is an equality whenever Σ is a round sphere. On the other hand, if equality holds in part (II), it follows that

$$\int_{\Sigma_t} H d\sigma = \left(\int_{\Sigma_0} H d\sigma\right) e^{\frac{n-2}{n-1}\cdot t} \text{ for a.e. } t \geq 0,$$

and therefore $H^2 = (n-1)|A|^2$ on Σ_t, for a.e. $t \geq 0$. This implies Σ_t is a disjoint union of round spheres, for a.e. $t \geq 0$. For a solution of inverse mean curvature flow in $\mathbb{R}^n$, this is only possible if Σ_t is, in fact, a single round sphere for every t. (See e.g. the Two Spheres Example 1.5 of [13].) This ends the poof of (II). □

Proof of Theorem 12 part (III). The transformation law for the mean curvature under conformal deformations is

$$H_g = u^{-\frac{2}{n-2}}\left(H_0 + \frac{2(n-1)}{n-2}\frac{u_\nu}{u}\right).$$

This together with the divergence theorem gives that

$$\begin{aligned}\int_{\mathbb{B}_\rho\setminus\Omega} \Delta_0 u dV_0 &= \int_{S_\rho} u_r d\sigma_\rho^0 - \int_\Sigma u_\nu d\sigma_0 \\ &= -m\omega_{n-1}\frac{n-2}{2} + O(\rho^{-1}) + \frac{n-2}{2(n-1)}\int_\Sigma H_0 u d\sigma_0.\end{aligned}$$

Taking the limit $\rho \to \infty$ we obtain

$$m = -\frac{2}{(n-2)\omega_{n-1}}\int_{\Omega^c} \Delta_0 u dV_0 + \frac{1}{(n-1)\omega_{n-1}}\int_\Sigma H_0 u d\sigma_0. \tag{3.6}$$

Since $\Delta_0 u \leq 0$ on Ω^c and $u \geq \alpha$ on Σ, this gives the inequality in (III).

Rigidity. For the rigidity statement of (III) we only need to prove one direction since (clearly) for the Riemannian Schwarzschild manifold, the above inequalities are all equalities. Here we may not assume that u is harmonic at infinity (although this will follow from the claim below).

If equality holds in (III), we have that

$$\int_\Sigma H_0 d\sigma_0 = (n-1)\omega_{n-1}\frac{m}{\alpha}. \tag{3.7}$$

Claim. u is harmonic on Ω^c, and is (the same) constant on (all components of) Σ.

Proof. By equations (3.6) and (3.7) it follows that u is harmonic on Ω^c. Note that since the inequality in (III) is obtained from equation (3.6) by replacing u by its minimum on Σ, it follows that, in the case of equality in (III), u equals its minimum on Σ, i.e. $u|_\Sigma \equiv \min_\Sigma u = \alpha$. □

Claim. $\alpha \geq 2$.

Proof. From the previous claim $\Delta_0 u = 0$, so

$$0 = \int_{\Omega^c} u\Delta_0 u dV_0 = -\int_{\Omega^c} |\nabla_0 u|^2 dV_0 - \frac{m}{2}\omega_{n-1}(n-2) - \int_\Sigma u u_\nu d\sigma_0.$$

Also, from that claim $u_{|\Sigma} \equiv \alpha$, so we know u is the optimal function for $C_0^{(\alpha,1)}(\Sigma)$ (cf. Remark 2). Furthermore, using Remark 3 it follows that $\int_{\Omega^c} |\nabla_0 u|^2 dV_0 = (n-2)\omega_{n-1}(\alpha-1)^2 C_0(\Sigma)$. Combining this with the above equation we obtain

$$(n-2)\omega_{n-1}(\alpha-1)^2 C_0(\Sigma) = -\frac{m}{2}\omega_{n-1}(n-2) + \frac{n-2}{2(n-1)}\alpha^2 \int_\Sigma H_0 d\sigma_0.$$

We now use equation (3.7) to substitute the last term in the above equation. We get

$$(\alpha-1)C_0(\Sigma) = \frac{m}{2}. \tag{3.8}$$

It is easy to see that equations (3.7), (3.8), combined with the inequality in (II), imply $\alpha \geq 2$. □

This concludes the proof of (III). □

Proof of Theorem 12 part (V). Recall that a hypersurface $\Sigma = \partial\Omega \subset \mathbb{R}^n$ is called **outer-minimizing** if whenever Ω' is a domain with $\Omega' \supset \Omega$ then $|\partial\Omega'| \geq |\Sigma|$. (An example of such a hypersurface is given by the boundary of a collection of sufficiently far-apart convex bodies in $\mathbb{R}^n$.) Let us denote by $|\Sigma_t|$ the area of the evolving hypersurface Σ_t moving by IMCF with initial condition $\Sigma_0 \equiv \Sigma$. Then, by Lemma 1.4 of [13], one has $|\Sigma_t| = e^t|\Sigma|$ for all $t \geq 0$, provided Σ is outer-minimizing.

Now, from Lemma 14 and the fact that $e^{(\frac{n-2}{n-1})t} = (|\Sigma_t|/|\Sigma|)^{\frac{n-2}{n-1}}$, we have that the function

$$f(t) := |\Sigma_t|^{-\frac{n-2}{n-1}} \int_{\Sigma_t} H d\sigma_t$$

is non-increasing along IMCF in $\mathbb{R}^n$. By a known property of Euclidean IMCF, for t large enough Σ_t is arbitrarily close to a round sphere, and hence $f(t) \to (n-1)\omega_{n-1}^{1/(n-1)}$ as $t \to \infty$. This proves the inequality in (V), since $f(0) = |\Sigma|^{-(n-2)/(n-1)} \int_\Sigma H d\sigma$.

Rigidity. From Remark 15 it follows that the inequality of part (V) is an equality whenever Σ is a round sphere. On the other hand, if the inequality in (V) were an equality, we have $f(\infty) = f(0)$, so $f(t) \equiv f(0)$ for all t since f is non-increasing. This implies $\int_{\Sigma_t} H d\sigma_t = ce^{t(n-2)/(n-1)}$, and inequality (3.4) becomes an equality. Thus, we have reduced rigidity here to the case of rigidity of part (II). □

4. Euclidean estimates

An interesting consequence of the work [9] (which is parts (II) and (V) of Theorem 12 above) is that we have obtained new proofs of the following classical Euclidean estimates *under weaker hypotheses.*

THEOREM 16 (Freire-Schwartz [9]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain (not necessarily connected) with mean-convex boundary $\Sigma = \partial\Omega$. Denote by V_0 its volume, and by $A_0,\ H_0 > 0$ the area and mean curvature of Σ, respectively. Then*

(a) **Polya-Szegö inequality:**

$$\int_\Sigma H_0 d\sigma_0 \geq (n-1)\omega_{n-1}\mathrm{Cap}_0(\Sigma),$$

with equality achieved if and only if Ω is a round ball.

(b) **Aleksandrov-Fenchel inequality:** *Assume further that Σ is outer-minimizing. Then*

$$\int_\Sigma H_0 d\sigma_0 \geq (n-1)\omega_{n-1}\left(A_0/\omega_{n-1}\right)^{(n-2)/(n-1)},$$

with equality achieved if and only if Ω is a round ball.

The novelty in Theorem 16 is that it applies to many new cases and in high dimensions, and the case of equality is fully characterized. Indeed, both the Polya-Szegö and the Aleksandrov-Fenchel inequalities were originally proved for convex hypersurfaces $\Omega \subset \mathbb{R}^3$. Guan-Li [11] generalized (b) for mean-convex *star-shaped* domains. Guan-Li's result, like the original version, requires Ω to be, topologically, a ball. Our version allows for more topological types. Indeed, it is easy to see that handlebodies (and disjoint unions of them) may be constructed to be mean-convex and outermost.

An important application of Theorem 16 is to strengthen Lam's proof of the Riemannian Penrose inequality for asymptotically flat graphs [16]. Part (b) of Theorem 16 can be used *as-is* to improve Lam's result replacing "convex boundary components" by the more general "Euclidean mean-convex, outer minimizing boundary components" in his proof of the inequality.

Another possible application of the above techniques is to use them for proving the conformally flat case of a Riemannian Penrose inequality conjectured by Bray in [4] for manifolds containing both closed minimal hypersurfaces and *zero area singularities.* Partial work on this direction was carried out by Jauregui [15] motivated by the work of the author in [20].

References

[1] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693.

[2] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267.

[3] H. L. Bray and K. Iga, *Superharmonic functions in $\mathbb{R}^n$ and the Penrose inequality in general relativity*, Comm. Anal. Geom. **10** (2002), no. 5, 999–1016.

[4] H. L. Bray and J. L. Jauregui, *A geometric theory of zero area singularities in general relativity* (2009), available at `arXiv:0909.0522[math.DG]`.

[5] H. L. Bray and D. A. Lee, *On the Riemannian Penrose inequality in dimensions less than eight*, Duke Math. J. **148** (2009), no. 1, 81–106.

[6] H. L. Bray and P. Miao, *On the capacity of surfaces in manifolds with nonnegative scalar curvature*, Invent. Math. **172** (2008), no. 3, 459–475.

[7] G. L. Bunting and A. K. M. Masood-ul-Alam, *Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time*, Gen. Relativity Gravitation **19** (1987), no. 2, 147–154.
[8] P. T. Chruściel, *A remark on the positive-energy theorem*, Classical Quantum Gravity **3** (1986), no. 6, L115–L121.
[9] A. Freire and F. Schwartz, *Mass-capacity inequalities for conformally flat manifolds with boundary*, Comm. Partial Differential Equations (to appear), available at `arXiv:1107.1407`.
[10] R. Geroch, *Energy extraction*, Ann. New York Acad. Sci. **224** (1973), 108–117.
[11] P. Guan and J. Li, *The quermassintegral inequalities for k-convex starshaped domains*, Adv. Math. **221** (2009), no. 5, 1725–1732.
[12] K. Hildén, *Symmetrization of functions in Sobolev spaces and the isoperimetric inequality*, Manuscripta Math. **18** (1976), no. 3, 215–235.
[13] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437.
[14] G. Huisken and T. Ilmanen, *Higher regularity of the inverse mean curvature flow*, J. Differential Geom. **80** (2008), no. 3, 433–451.
[15] J. L. Jauregui, *Penrose-type inequalities with a Euclidean background*, available at `arXiv:1108.4042v1`.
[16] M-K. G. Lam, *The graph cases of the Riemannian positive mass and Penrose inequalities in all dimensions*, ArXiv (20 Oct 2010), 101.4256v1[math.DG].
[17] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J., 1951.
[18] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.
[19] R. Schoen and S. T. Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** (1981), no. 2, 231–260.
[20] F. Schwartz, *A Volumetric Penrose Inequality for Conformally Flat Manifolds*, Ann. Henri Poincaré **12** (2011), 67–76.
[21] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.
[22] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402.

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