“Reader’s manual” for MA4A2 Advanced Partial Differential Equations (Reading Course)

F. Theil & T.J. Sullivan

October 1, 2007

Course administration

- Course leader/moderator: Dr Florian Theil, f.theil@warwick.ac.uk.
- Teaching assistant: Tim Sullivan, t.j.sullivan@warwick.ac.uk.
- Classes: 2 hours per week in term 1 (see below for details).
- Credit: 18 CATS.
- Examination: 3 hour written exam in April 2008 (term 3).

MA4A2 Advanced PDEs is a reading course: this document lays out the topics that comprise the examinable syllabus. There are also some additional topics, indicated with double daggers ‡like this‡; you should try to understand them even though they are more technically complicated and non-examinable. You should aim to learn the material by (a) reading the recommended texts or any others that you may find helpful, and (b) since mathematics is not a spectator sport, doing as many of the exercises as possible. Some exercises will be provided in a separate document; try any other exercises in the recommended texts that you find helpful. If you can comfortably solve all of the exercises and questions from past exams, then you will be in good shape for this year’s exam.

There will be two one-hour support classes per week:

- Tuesdays, 1000–1100 in room MA-B0.13: this will be a traditional support class, discussing the week’s topics and associated exercises;
- Tuesdays, 1100–1200 in room MA-B0.13: this session will focus on the exercises relevant to the week’s topics — the emphasis will be on *students presenting their solutions to each other*.

Prerequisites

- Measure theory, e.g. **MA359 Measure Theory**: L^p spaces; Hölder’s inequality, Young’s inequality & c.
- Linear functional analysis, e.g. **MA3F4 Linear Analysis**: Banach and Hilbert spaces; completeness, compactness, reflexivity, separability & c.; Riesz representation theorem; Banach-Alaoglu weak-$*$ compactness theorem; Fredholm alternative for compact operators on Hilbert spaces.

- Classical theory of PDEs, e.g. **MA3G1 Theory of Partial Differential Equations**: helpful as background material, but not essential. MA3G1 covers the "classical", pre-1950 theory of PDEs, whereas this course is grounded in the modern functional-analytic approach.

See [Ev, appendices] for a review of some of these areas.

Week 1 Review and some basics

- What is a partial differential equation? Linear/semi-linear/quasi-linear PDEs. [Ev, §1.1, appendix A] [RR, §2.1.4] [SS, §0]
- Notations for partial derivatives; multi-indices. [Ev, §1.5] [SS, §0]
- Laplace's equation. Harmonicity and the mean value property. [Ev, §2.2] [SS, §1.1]
- The fundamental solution; the Green's function. [Ev, §2.2] [SS, §1.2]

Week 2 Sobolev spaces

Note that the Taught Course Centre in Mathematics, a joint Bath-Bristol-Imperial-Oxford-Warwick initiative, will be offering a course on Sobolev spaces [Bu] by Prof. Geoffrey Burton of the University of Bath. Postgraduate students may take this course for credit; undergraduate students may not. Regardless of under-/post-graduate status, this course may serve as a useful additional reference for weeks 2–5 of MA4A2, but will *not* cover weeks 6–10.

- Weak differentiability of functions $u\colon \Omega \to \mathbb{R}$ in terms of $C_c^\infty(\Omega)$ test functions, Ω open in $\mathbb{R}^n$. Properties of the weak derivative. [Ev, §5.2.1, §5.2.3] [RR, §5.2] [Sh, §2.4] [SS, §2.1]
- Sobolev spaces $W^{k,p}(\Omega)$ [Ev, §5.2.2] [RR, §6.4.1] [Sh, §2.4] [SS, §2.1]; weak differentiability in terms of test functions vs. difference quotients [Ev, §5.8.2] [RR, §8.5.1].
- The Fourier transform of $u \in H^k(\mathbb{R}^n)$. ‡$H^s(\mathbb{R}^n)$ for non-integral $s > 0$.‡ [Ev, §5.8.4] [RR, §6.4.2]
- (Equivalent) norms for $W^{k,p}(\Omega)$; completeness and separability [RR, §6.4.1] [SS, §2.1]; consequences *via*, e.g., Banach-Alaoglu theorem [SS, §2.3].
- $W^{k,p}_{\mathrm{loc}}(\Omega)$-approximation of $u \in W^{k,p}(\Omega)$ by smooth functions "ε-away from $\partial\Omega$". [Ev, §5.3.1] [Sh, §2.4] [SS, §2.2]
- Density of smooth functions with compact support [Ev, §5.2.2–3]; the space $W^{k,p}_0(\Omega)$ [Ev, §5.2.2] [RR, §6.4.1] [Sh, §2.4] [SS, §2.2].

Week 3 More Sobolev spaces

- Extension of $u \in W_0^{k,p}(\Omega)$ by 0 outside Ω. [RR, §6.4.5, case $p = 2$] [SS, §2.2]
- Friedrichs' inequality; equivalent norms for $W_0^{k,p}(\Omega)$. [SS, §2.2]
- Approximation up to $\partial\Omega$ of $u \in W^{k,p}(\Omega)$ by smooth functions for (star-shaped) C^1 Ω. [Ev, §5.3.3] [SS, §2.4]
- Extension operator $E \colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ [Ev, §5.4] [RR, §6.4.5] [SS, §2.5]. ‡Extension operator $E \colon W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$.‡ [SS, §2.5]
- ‡Trace operator $T \colon W^{k,p}(\Omega) \to W^{k-1,p}(\partial\Omega)$; $u \in W_0^{k,p}(\Omega) \iff u \in W^{k,p}(\Omega)$ and $Tu = 0$; applications e.g. Green's theorem.‡ [Ev, §5.5] [RR, §6.4.8] [SS, §2.6]

Week 4 Embeddings and inequalities

- Continuous and compact embeddings/inclusions. [Ev, §5.7] [RR, §6.4.7] [SS, §2.8]
- Gagliardo-Nirenberg inequality for $W^{1,p}(\mathbb{R}^n)$, $W^{1,p}(\Omega)$: continuous embedding $W^{1,p} \hookrightarrow L^{p^*}$ for $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. [Ev, §5.6.1] [SS, §2.8]
- Continuous embedding $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\frac{1}{q} = \frac{1}{p} + \frac{k}{n}$, $kp < n$. [Ev, §5.6.3] [SS, §2.8]
- Poincaré's inequality. [Ev, §5.6.1, §5.8.1] [RR, §6.4.7] [SS, §2.8]
- Rellich-Kondrachov theorem: compact embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ for $1 \le q < p^*$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. [Ev, §5.7] [Sh, §2.4] [SS, §2.8]

Week 5 More embeddings and inequalities

- Hölder spaces $C^{k,\alpha}(\bar{\Omega})$. [Ev, §5.1] [SS, §2.9]
- Morrey's inequality for $W^{1,p}(\mathbb{R}^n)$, $W^{1,p}(\Omega)$: continuous embedding $W^{1,p} \hookrightarrow C^{0,\alpha}$ for $p > n$, $\alpha = 1 - \frac{n}{p}$. [Ev, §5.6.2] [SS, §2.9]
- ‡Boundary case $kp = n$; Orlicz and Orlicz-Sobolev spaces.‡ [SS, §2.9]
- The dual space $H^{-s}(\Omega)$ [Ev, §5.9.1] [RR, §6.4.9] [SS, §2.11]; representation of $\ell \in H^{-1}(\Omega)$ [Ev, §5.9.1] [SS, §2.11].

Week 6 Elliptic PDEs

- Elliptic operators and equations; adjoint operator; the associated bilinear form; weak solutions. [Ev, §6.1.1–2, §6.2.3] [RR, §8.2.1] [SS, §3.1]
- Lax-Milgram theorem. [Ev, §6.2.1] [RR, §8.2.2] [Sh, §1.2] [SS, §3.1]

- H^1_0-bounds (energy estimates) for the bilinear form (i.e. Gårding's inequality for $k = 1$) [Ev, §6.2.2] [RR, §8.2.3] [SS, §3.1]; consequences for existence and uniqueness of weak solutions. [Ev, §6.2.2] [SS, §3.1]

- Fredholm alternative for elliptic operators. [Ev, §6.2.3] [RR, §7.5.1] [SS, §3.1]

Week 7 Some spectral theory for elliptic operators

- Spectrum of an elliptic operator; basic properties. [Ev, §6.2.3] [SS, §3.1]

- Solutions to the eigenvalue problem and boundedness of the inverse outside the spectrum. [Ev, §6.2.3] [SS, §3.2]

- Hilbert-Schmidt theorem for symmetric elliptic operators ‡and compact self-adjoint operators.‡ [RR, §7.5.1] [SS, §3.2]

- The principal eigenvalue and Rayleigh's formula for symmetric elliptic operators. [Ev, §6.5.1] [SS, §3.2]

Week 8 Parabolic PDEs

- Parabolic operators and equations; weak solutions. [Ev, §7.1] [SS, §4.1]

- Bochner spaces, e.g. $L^p([0,T]; X)$, X a Banach space. [Ev, §5.9.2] [SS, §4.1] [Sh, §3.1]

- Galerkin approximations (also applicable to elliptic case) [Ev, §7.1.2] [RR, §9.3.2] [SS, §4.2]; Céa's inequality [exercises].

- Uniqueness of weak solutions using Grönwall's inequality. [Ev, §7.1.2] [SS, §4.2]

Week 9 More parabolic PDEs

- Lax-Milgram theory for parabolic PDEs on non-cylindrical domains, linear case [Sh, §3.2]; ‡degenerate and non-linear case [Sh, §3.3–4]‡

Week 10 Methods for non-linear problems

See the course **MA4G6 Calculus of Variations** for a more general treatment of variational methods; for this course, aim to understand only specific examples.

- Variational methods for the Poisson equation in $H^1_0(\Omega)$; the p-Laplace equation in $W^{1,p}_0(\Omega)$; the non-linear Poisson equation in $H^1_0(\Omega)$; the Poisson equation with von Neumann boundary conditions in $H^1(\Omega)$. [Ev, §8.1.2, §8.2] [SS, §2.3, §6.1]

- Monotonicity methods. [Ev, §9.1] [RR, §9.3.3]

References

[Bu] Burton, Geoffrey R. (2007–08) *MA60207 Sobolev Spaces.* Taught Course Centre in Mathematics. http://www.maths.bath.ac.uk/~grb/Sobolev/frontpage.html

[Ev] Evans, Lawrence C. (1998). *Partial differential equations.* Providence, RI: American Mathematical Society. ISBN 0-8218-0772-2.

[RR] Renardy, Michael and Rogers, Robert C. (2004). *An introduction to partial differential equations*, Second edition, Texts in Applied Mathematics 13, New York: Springer-Verlag. ISBN 0-387-00444-0.

[Sh] Showalter, Ralph E. (1997) *Monotone operators in Banach space and nonlinear partial differential equations.* Mathematical Surveys and Monographs 49. Providence, RI: American Mathematical Society. ISBN 0-8218-0500-2

[SS] Slastikov, Valeriy and Sullivan, Timothy J. (2006) *Lecture notes for MA4A2 Advanced PDEs.* http://www.tjsullivan.org.uk/math/MA4A2.pdf.