MIDTERM #1 MATH 32 – FALL 2005

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Points: /100

I have adhered to the Duke Community standard in completing this test.

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signature

Here's a list of formulas that may be of use to you. *Good luck!*

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- Error bound for Trapezoidal Approximation is $|ET_n| \leq \frac{M(b-a)^3}{12n^2}$

1. (a) (3pts) Write down the definition of the integral of $f(x)$ over the interval $[a, b]$ in terms of a Riemann sum.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

(b) (4pts) Use the definition of part (a) to compute the following integral:

$$\int_0^1 \left(\frac{x}{3}+1\right) dx$$

2 pts. $\int_0^1 (\frac{x}{3}+1)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{x_i}{3}+1)\,\Delta x$, $\Delta x = \frac{1}{n}$, $x_i = \frac{i}{n}$

1 pt. $= \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{1}{3}\frac{i}{n}+1)\frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{3}\frac{i}{n^2}+\frac{1}{n}\right)$

1 pt. $= \lim_{n\to\infty} \left[\frac{1}{3n^2}\left(\sum_{i=1}^{n} i\right)+1\right] = \lim_{n\to\infty} \left[\frac{1}{3n^2}\,\frac{n(n+1)}{2}+1\right]$

$= \frac{1}{6}+1 = 7/6$

(c) (3pts) Compute the integral of (b) using a direct method.

$$\int_0^1 (\frac{x}{3}+1)\,dx = \left[\frac{x^2}{6}+x\right]_0^1 = \frac{1}{6}+1 = 7/6$$

2. Compute the following limits by expressing them as a definite integral over $[0,1]$ and then evaluating the integral.

(a) (4pts)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}$$

$= \lim_{n\to\infty} \sum_{i=1}^{n} 1\cdot\frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} 1\cdot\Delta x$

$= \int_0^1 dx = [x]_0^1 = 1$.

(b) (4pts)

$$\lim_{n\to\infty} \frac{n + 2n + 3n + 4n + \ldots + n^2}{n^3}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{in}{n^3} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i \cdot \Delta x = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

(c) (4pts)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{10}{n^2} \left(i\sqrt{3 + 5\frac{i^2}{n^2}} \right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{10i}{n} \sqrt{3+5\left(\frac{i}{n}\right)^2} \right) \frac{1}{n} = \int_0^1 10x\sqrt{3+5x^2}\,dx$$

$$\begin{pmatrix} u = 3+5x^2 \\ du = 10x\,dx \end{pmatrix} \quad \stackrel{\downarrow}{=} \int \sqrt{u}\,du = \frac{2}{3} u^{3/2} = \left[\frac{2}{3} (3+5x^2)^{3/2} \right]_0^1$$

$$= \frac{2}{3}\left(8^{3/2} - 3^{3/2}\right).$$

3. (a) (4pts) Compute the derivative of

$$f(x) = \int_{-2}^{x} \ln(\sin(t) + 2)dt$$

$$f'(x) = \ln(\sin(x) + 2) \quad (\text{FTC})$$

(b) (4pts) The average value of the function $f(t)$ on the interval $[0, x^2]$ is $\frac{1}{x}$ for all $x > 0$. Find $f(x)$.

1 pt. $\left\{ \text{Avg} = \frac{1}{b-a}\int_a^b f(x)\,dx \Rightarrow \frac{1}{x^2}\int_0^{x^2} f(t)\,dt = \frac{1}{x} \right.$

2 pts. $\left\{ \int_0^{x^2} f(t)\,dt = x \;\; / \frac{d}{dx} \Rightarrow f(x^2)\,2x = 1 \right.$

1 pt. $\left\{ f(x^2) = \frac{1}{2x} \quad \text{so} \quad f(x) = \frac{1}{2\sqrt{x}}. \right.$

(c) (4pts) Compute the derivative of

$$f(x) = \int_0^x \left(\int_0^t y\,dy\right) dt$$

$$f'(x) = \int_0^x y\,dy = \left[\frac{y^2}{2}\right]_0^x = \frac{x^2}{2}.$$

(d) (4pts) Compute the derivative of

$$f(x) = e^{\int_1^x ln(t)dt}$$

$$f'(x) = e^{\int_1^x \ln(t)\,dt} \cdot \underbrace{\ln(x)}_{\left(= \frac{d}{dx}\int_1^x \ln(t)\,dt\right)}$$

(chain rule)

4. (a) (8pts) Compute the volume obtained by rotating around the x-axis the region in bound by the curves $y_{out}(x)$, $y_{in}(x)$ between $x = 0$ and $x = 2/3$, where

$$y_{out}(x) = \sqrt{\frac{2x}{x^2 - x^3 + 3}}, \quad y_{in}(x) = \sqrt{\frac{3x^2}{x^2 - x^3 + 3}}$$

3 pts $\left\{ V = \int_0^{2/3} (\pi y_{out}^2 - \pi y_{in}^2)\, dx = \pi \int_0^{2/3} \left(\frac{2x}{x^2 - x^3 + 3} - \frac{3x^2}{x^2 - x^3 + 3} \right) dx \right.$

2 pts $\left\{ = \pi \int_0^{2/3} \frac{2x - 3x^2}{x^2 - x^3 + 3}\, dx \right.$

3 pts $\left\{ = \left[\pi \ln|x^2 - x^3 + 3| \right]_0^{2/3} \right.$

$= \pi \left[\ln\left(\frac{4}{9} - \frac{8}{27} + 3\right) - \ln(3) \right]$

(b) (8pts) Let R be the region on the xy-plane that is bounded by the curves $y_1(x) = x + \sqrt{x}$, $y_2(x) = x^2 + \sqrt{x}$. Compute the volume of the solid whose base is the region R and whose cross-section perpendicular to the x-axis is a square.

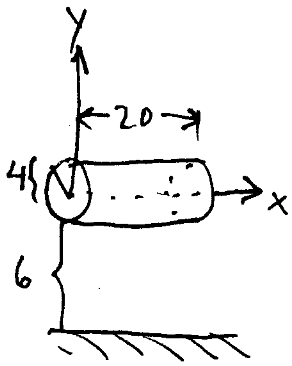
2 pts $\left\{ y_1(x) = y_2(x) \iff x + \sqrt{x} = x^2 + \sqrt{x} \iff x = x^2 \quad \therefore x = 0, 1 \right.$

4 pts $\left\{ V = \int_0^1 [(x + \sqrt{x}) - (x^2 + \sqrt{x})]^2\, dx = \int_0^1 (x - x^2)^2\, dx \right.$

2 pts $\left\{ = \int_0^1 (x^2 - 2x^3 + x^4)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right]_0^1 \right.$

$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$

5. The firefighter airplane *Skippy Jr.* has a cylindrical water tank that lays horizontally inside its belly. The receptacle is $20ft$ long and has a radius of $4ft$. When the plane is parked, the bottom of the tank is $6ft$ above ground.



(a) (4pts) Assume that the density of water is 62.4 lb/ft^3. Write down an integral (DO NOT COMPUTE IT!) that expresses the amount of work required to fill the plane's water tank pumping from a stream that passes right under the plane.

$$W = \int_{-4}^{4} (y+10)\, \rho\, A(y)\, dy$$

$$x = \sqrt{16-y^2}, \quad A(y) = 2x \cdot 20 = 40\sqrt{16-y^2}$$

$$W = \int_{-4}^{4} (y+10)(62.4)\, 40\sqrt{16-y^2}\, dy$$

(b) (4pts) Assume that the plane weighs $10{,}000lb$ when the water tank is empty. How much work is required from *Skippy Jr.* to go from sea level to $5{,}000ft$ with a full water reservoir?

$$\text{Tank weight} = \rho \cdot \text{Volume} = (62.4)(16\pi \cdot 20)$$

$$W = \text{weight} \cdot \text{distance} = 5{,}000\Big(\underbrace{10{,}000}_{\text{plane}} + \underbrace{(62.4)(16\pi \cdot 20)}_{\text{water tank}}\Big)$$

(c) (4pts) Suppose that the water tank has a leak and water pours out at a rate of $10lb/s$. Assume *Skippy Jr.* ascends with constant vertical speed of $100ft/s$. Write down an integral (DO NOT COMPUTE IT!) that expresses the amount of work done by *Skippy Jr.* to go from sea level to $5{,}000ft$ starting with a full water reservoir.

$$W = \int_0^{50} \left[10{,}000 + (62.4 \cdot 16\pi \cdot 20) - 10t\right] dt$$

or

$$W = \int^{5000} \left[10{,}000 + (16\pi \cdot 20 \cdot 62.4) - y\right] dy$$

6. (a) (3pts) Write down the integral formula to that expresses the length L of the curve $y = f(x)$ between $x = a$ and $x = b$.

$$L = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

(b) (4pts) Compute the length of the curve

$$y = (x - \frac{4}{9})^{3/2}$$

between $x = 1$ and $x = 2$.

$$L = \int_1^2 \sqrt{1 + \left[\frac{3}{2}(x - \frac{4}{9})^{1/2}\right]^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{9}{4}x - 1}\, dx = \frac{3}{2}\int_1^2 \sqrt{x}\, dx$$

$$= \left[\cancel{\tfrac{3}{2}}\left(x^{3/2} \cdot \tfrac{2}{\cancel{3}}\right)\right]_1^2 = 2^{3/2} - 1.$$

(c) (3pts) Write down the integral formula that expresses the surface area obtained by rotating the curve $y = f(x)$ around the x-axis, between the values of $x = a$ and $x = b$.

$$A = \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

(d) (4pts) Set up the most elementary integral (DO NOT COMPUTE IT!) that expresses the surface area of the solid obtained by rotating the curve

$$x = \frac{1}{12}y^6 + \frac{1}{8}y^{-4}$$

about the x-axis between the values of $y = 1$ and $y = 2$.

3 pts

$$\frac{dx}{dy} = \frac{1}{2}y^5 - \frac{1}{2}y^{-5}$$

$$A = \int 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_1^2 2\pi y\sqrt{1+\frac{1}{4}y^{10}-\frac{1}{2}+\frac{1}{4}y^{-10}}\,dy$$

1 pt

$$= \int_1^2 2\pi y\left(\frac{1}{2}y^5+\frac{1}{2}y^{-5}\right)dy$$

7. (8pts) Determine how large n has to be in the trapezoidal approximation T_n of

$$I = \int_2^3 \frac{1}{x-1}\,dx$$

in order to get an error $|ET_n|$ smaller than .01.

4 pts

Want $\frac{M(b-a)^3}{12n^2} \leq \frac{1}{100}$;

$|f''(x)| \leq M \quad a \leq x \leq b$

$f(x) = \frac{1}{x-1} = (x-1)^{-1}$

$f'(x) = -(x-1)^{-2}$

$f''(x) = 2(x-1)^{-3}$

So

$$\frac{2}{(x-1)^3} \leq \boxed{2 = M}$$

4 pts

$$\frac{2(1)^3}{12n^2} \leq \frac{1}{100} \Rightarrow n^2 \geq \frac{100}{6}$$

$$\therefore \boxed{n \geq 5}$$

8. Let $f(x) = \frac{\ln(x)+1}{x}$ be a function defined for $x > 1$. (We say in this case that the domain of f is $(1, \infty)$.)

(a) (4pts) Use a fact from class to explain why does f has an inverse function g.

$$f'(x) = \frac{x\frac{1}{x} - (\ln(x)+1)\cdot 1}{x^2} = -\frac{\ln(x)}{x^2} < 0 \quad \text{for } x>1.$$

Hence $f'(x) \neq 0$ & then it has an inverse.

(b) (4pts) Find the [redacted] range of the inverse function g.

$$\text{range}(g) = \text{domain}(f) = (1, \infty)$$

(c) (4pts) Find $g(\frac{2}{e})$.

want x s.t.

$$\frac{\ln(x)+1}{x} = \frac{2}{e}.$$

Since $\ln(e) = 1$, one sees

$$\frac{\ln(e)+1}{e} = \frac{2}{e}, \text{ so } \boxed{g\left(\frac{2}{e}\right) = e}$$