An overview to digest what we have done so far before exam 1

For PDE’s (unlike ODE’s) we have more or less separate theories for various kinds of PDEs. For each kind, the theory should answer the following questions:

- What are meaningful additional conditions (initial, or boundary conditions) by which to complement the PDE such as to get a problem that has one and only one solution?
- Prove existence and uniqueness. In doing so, identify qualitative hypotheses on the data that assert such existence and uniqueness (e.g., the data should be continuous functions).
- For special situations (only), one may have a chance for a ‘solution formula’. Establish such formulas, where and inasmuch possible.

Data for a PDE come in different kinds:

- geometric data about the domain in which we study the PDE; we mainly have the following cases: full space, half space, and ball. They give more explicit results. Other cases are bounded domains with some ‘decent’ boundary. The geometrically special cases which we studied in more detail will later serve as either prerequisite tool or motivation for results about more general domains.
- given functions entering into boundary or initial conditions, or into the PDE itself. For them, we usually need to make somewhat plausible hypotheses concerning boundedness, continuity, or smoothness (sometimes compact support). Which of these hypotheses we need to make depends on the particular problem at hand. We usually strive for reasonably general hypotheses, but will not invest big effort to get the most general hypotheses.

We have made first steps into this program for three types of equations:
(A) the transport equation (very little so far)
(B) the Laplace and Poisson equations
(C) the heat equation (just begun)

There are certain tools and themes that recur in this quest to understand various types of PDEs:

(1) **Reduction to an ODE when possible.**

For the transport equation, this idea saved the day: There are simple lines in the \((x, t)\) plane (or space) along which the PDE reduces to an ODE: In a PDE \(u_t + b \cdot \nabla u = f(x, t, u)\) (we actually studied only special cases of \(f\)), the partial differential operator \(\partial_t + b \cdot \nabla\) is a directional derivative in a certain direction in \((x, t)\) space, namely the direction \((b, 1)\). So if we restrict the unknown function \(u\) to lines that go in exactly this direction, then the restricted function satisfies an ODE along this line.

For the Laplace equation, we could find a radial solution that turned out to be the ‘fundamental solution’. Since we assumed radial symmetry, we again reduced the PDE to an ODE; but this time, we only obtained very special solutions.
(2) **Integration by parts and/or the divergence theorem.**

For the transport equation, we used integration by parts to capture in a mathematically rigorous way the idea that, from the physical interpretation, certain functions should be considered as valid solutions to the PDE even though they do not have as many derivatives as the PDE requires at face value. This led to the notion of a weak solution, which we did not pursue at more depth for the time being, but which you should understand at the level of the homework problem dedicated to it. The idea of a weak solution is that we multiply the PDE with a smooth function of compact support (called test function) and integrate. In the integral expression, we use integration by parts to move all (or some) derivatives to the test function. This integrated formulation is equivalent to the original PDE for smooth solutions, but generalizes the solution concept if the smoothness hypothesis is relaxed.

For the Laplace equation, we did not discuss a notion of weak solution: it is (almost) not needed, because solutions turn out to be automatically smooth. We did however use the divergence theorem in the version of Green’s formula as a tool to analyze the structure of solutions to the Poisson equation and the Laplace equation.

(3) **Smooth functions with compact support.**

They belong in modern calculus and have a-priori nothing to do with PDEs; but they are a useful tool for PDEs. We have used them in various contexts. Convolution with such functions allows to approximate a ‘rough’ function by a smooth function. We did this to get a smooth partition of unity form a merely continuous one, when we found such a partition of unity useful to prove Gauss’ divergence theorem for reasonably general domains (with $C^1$ or piecewise $C^1$ boundary). We also used the smoothing effect of a convolution to prove that a continuous function that satisfies the mean value property for harmonic functions is actually smooth (in particular $C^2$) and therefore harmonic.

We used such functions to show that, if $\int f \phi = 0$ for every smooth $\phi$ with compact support (and $f$ is continuous), then $f$ must be 0. This simple argument was used first in connecting weak solutions of the transport equation with classical solutions; it was also used to get the PDE out of the minimization property from Dirichlet’s principle.

(4) **The delta distribution.**

‘Distribution’ refers to a generalization of the notion of ‘function’ that we have no point to discuss here in detail. While a certain ‘theory of distributions’ is very useful in modern PDEs, we merely studied an informal notion of the ‘delta distribution’, which is viewed in physics like a density of a particle whose unit mass is concentrated in a single point. Since we refrained from introducing a general theory of distributions that would include the delta distribution as a special case, we had the poor-man’s version of capturing delta rigorously, namely the notion of a Dirac sequence. This notion was around in most cases where a ‘delta distribution idea’ and the idea that a certain sequence of functions ought to converge to delta had to be made formally rigorous.

In the examples with fundamental solution and Green’s function, where the idea that “the Laplacian of these functions should be $-\delta$” needed to be captured rigorously, the limiting process being used was not a Dirac sequence of functions, but rather a sequence of domains with little balls cut out around the singularity.

The (informal) claim $g(x) = \in \int g(y) \delta(x - y) dy$ means that any continuous function can be written as a ‘continuous superposition’ of $\delta$ distributions. If our PDE is linear, the superposition principle applies; and solutions for the special case where some data are
a δ distribution, become building blocks for the general case. These building block solutions can often be found explicitly, in a way that does not even require any knowledge about δ. The superposition formula can be motivated and made intuitive by a non-rigorous appeal to δ, but will eventually be proved directly, without distribution theory. Instead, the tools mentioned above will be used. So the rigor of these formulas is not compromised by any vagueness in the notions motivating them.

Laplace and Poisson Equation

Specifically, for the Laplace equation and the Poisson equation, we achieved the following:

- We found a function that is harmonic in punctured space and turns out to be the fundamental solution. By fundamental solution, we mean a solution in all of space that satisfies (informally speaking) the equation \(-\Delta u = \delta(x)\). By superposition of translates of the fundamental solution, we can obtain solutions to \(-\Delta u = f(x)\), say, for \(f \in C^2_{cpt}\) (which is by far not the most general hypothesis). Short of growth hypotheses, there can be no uniqueness theorem on the whole space. In dimension 3 or above, it is reasonable to ask for *bounded* solutions. The formula that amounts to a superposition of fundamental solutions for \(f \in C^2_{cpt}\) provides such a bounded solution. The fundamental solution in 2 dimensions is unbounded near infinity, and so we cannot generally expect bounded solutions to \(-\Delta u = f\) in 2 dimensions either.

- We found the mean value property for harmonic functions: a harmonic function equals, in each point, its average over a ball, or over a sphere centered at that point. This property had very strong consequences: It provided a maximum principle from which, in turn, we can obtain uniqueness of a solution to a boundary value problem on a bounded domain. It also guarantees that harmonic functions are smooth; in fact analytic. Even mere continuity plus the mean value property suffices to show harmonicity. These smoothness estimates can be made quantitative. For harmonic functions, they give bounds for the derivatives in a smaller domain in terms of the function values in a larger domain. This control over the derivatives implies Liouville’s theorem, which states that a function that is harmonic in the whole space, must be a constant. Liouville’s theorem amounts to a modified uniqueness for solutions to \(-\Delta u = f\) in the whole space: *Bounded* solutions to such a problem are unique up to constants.

- As a side, a mean value property in the form of an inequality rather than an equality is also available for subharmonic functions. We defined a subharmonic function to be \(C^2\) and to satisfy the inequality \(-\Delta u \leq 0\). However, a generalized notion (continuous and satisfying the inequality version of the mean value property) is also useful and will play a role in a later existence proof for the harmonic interpolation problem. We used the word ‘subsolution’ (to a BVP for the harmonic equation) for this notion.

- Another consequence of the mean value property was the Harnack inequality for *positive* harmonic functions in a domain: There are positive constants depending only on the domain \(U\) in which \(u > 0\) is harmonic, and a subdomain \(V\) compactly contained in \(U\) (this info includes the dimension), such that the ratio between any two values of \(u\) in \(V\) is bounded above and below by these constants. The estimate does not depend on the function \(u\) (otherwise it would be useless). While the precise value of such constants is pretty unspecified, it would turn out in more advanced applications that such
a precise value is not really needed. The introductory nature of the present material
alas prevented us from giving a really convincing application of Harnack’s inequality.

• We studied the harmonic lifting (or harmonic interpolation) problem: Given a con-
tinuous function on the boundary of a domain $U$, find a harmonic function inside this
domain that has the given function as boundary data. In a ball, and also in half-space,
this problem could be solved by means of Poisson’s formula (an explicit integral for-
mula). We have not found a similar formula for other domains (and cannot expect a
simple formula for general domains). But we sketched some ideas how to construct a
solution to the harmonic interpolation problem in bounded domains as a supremum of
subsolutions, based on our knowledge how to solve the same problem in a ball.

This (sketched) result is very important in itself, but also gives the weight of wide
applicability to the Green’s function formalism.

• Our study of the harmonic interpolation problem was actually intertwined with a study
of the Poisson equation, and for the beginner, the meandering route of reasoning may
seem confusing at first. Let’s recapitulate it:

For any bounded domain with boundary good enough for the divergence theorem to
hold, we try to devise (kind of) a formula for a solution to $-\Delta u = f$ inside, $u = g$
on the boundary, *assuming* existence of a solution. The method of choice is Green’s
function (whose existence is likewise to be established yet, at this point). The basic idea
is to test the Poisson equation with the fundamental solution (centered at an arbitrary
point inside the domain) and use integration by parts to move the derivatives onto
the fundamental solution. This will turn $\int (\Delta u) \Phi$ into $\int u \Delta \Phi = - \int u \delta = -u$ (plus
boundary terms) and thus give an expression for $u$. In technical rigor, we cut out a ball
around the singularity of the fundamental solution. We recover the Dirac distribution
style feature of getting $u$ from under the integral, when we take the limit as the ball
shrinks to a point, from the extra boundary of the cut-out ball.

This idea alone does not work yet, because the boundary terms involve both the known
boundary values of $u$ and the (unknown) normal derivative of $u$. So instead we want to
modify the fundamental solution by subtracting a harmonic function designed in such
a way that the difference has values zero on the boundary. This modified function is
Green’s function. Now, if we test the PDE $-\Delta u = f$ with the Green’s function, the
unknown normal derivatives of $u$ are killed by the vanishing of Green’s function on the
boundary.

At this moment we have achieved not yet a solution, but a crucial reduction of the prob-
lem: If only we can solve the harmonic interpolation problem for all those boundary
values that come from the (shifted) fundamental solution, then we get a solution for-
ma for Poisson’s equation with arbitrary continuous boundary data (still expressing
a solution whose existence has been assumed rather than proved).

This (tentative) integral formula already reveals another interesting connection: the
right hand side $f$ needs to be integrated against Green’s function, whereas the boundary
data need to be integrated against the normal derivative of Green’s function (which we’ll
call the Poisson kernel). The naming is weird, since the Poisson kernel refers to dealing
with boundary data $g$, whereas in Poisson’s equation, the name Poisson refers to a right
hand side $f$ that will be dealt with by means of Green’s function. But we have to live
with the names as they are.
Now that we know exactly how the formulas ought to be if a solution exists, we can begin to wrap up the problem: By means of a slightly ingenious reflection method, the harmonic corrector function (and hence Green’s function) can be obtained out of the fundamental solution for two special domains: the half space and the ball. For these two cases, we now have an explicit solution formula, still assuming existence. Next we can check that the function found by this formula indeed provides a solution. This we did in the case \( f = 0 \) (i.e., for Laplace’s equation only). The reason for this restriction is technical ease. A deeper study would show that the formula is perfectly good to solve Poisson’s equation (general \( f \)) as well. The technical difficulty stems from the fact that mere continuity of \( f \) is not a sufficient hypothesis for the existence of a \( C^2 \) solution (you have seen an explicit example in the hwk), and instead a strengthened hypothesis called Hölder continuity would be required (\( f \in C^1 \) would be a sufficient hypothesis, but proving the result under this stronger hypothesis would give a misleading impression about the nature of the difficulty).

The proof that the representation formula solves the harmonic interpolation problem for the ball with continuous data relies on the idea that the Poisson kernel \( K(x, y) \) converges to a delta distribution \( \delta_{x^*}(y) \) concentrated at \( x^* \), as \( x \) converges to the boundary point \( x^* \). This is the idea is behind the proof in the book, and it is also guiding the slight variant of the proof I gave in class: we introduced the notion of a Dirac sequence of functions.

Finally, we could (but dodged it) give an abstract existence proof for harmonic interpolation in decent domains. We outlined two ideas how this can be done: Either by constructing the harmonic interpolation as a supremum of subsolutions, or an abstract existence proof via calculus of variations, based on Dirichlet’s integral (energy method). This latter requires advanced analysis like Lebesgue integral and Sobolev spaces, even though the non-technical part of the ideas is much easier to grasp.

- Dirichlet’s principle describes solutions to a BVP for Poisson’s equation as (absolute) minima of the energy integral. So we are actually dealing with a minimum problem where the variable is a function rather than a vector or number (as was the case in calculus). The notion ‘minimum implies PDE’ is nothing but a variant of the calculus principle that at a minimum of a quantity, the (directional) derivatives of that quantity in all directions have to vanish. The converse direction is less general, but relies on the fact that the energy integral depends *quadratically* (more general: in a convex way) on the unknown function.

**Heat Equation**

Now that we have begun to study the heat equation, let me outline this as well. There are some analogies in the treatment of Laplace equation and the heat equation.

- We have identified a fundamental solution, even though finding it was not so well-motivated; so instead we identified it without motivation and proved its properties. It leads to a representation formula for solutions to the inhomogeneous heat equation in the full space with arbitrary (bounded and continuous) initial data. Again, a Dirac sequence style argument shows that the claimed formula satisfies the initial condition, and the PDE is satisfied basically because the fundamental solution satisfies the PDE with delta distribution initial conditions.
Duhamel’s principle of dealing with an inhomogeneity for the heat equation is very reminiscent of the variation of parameters method from ODEs. It is not specific to the heat equation, but is a consequence of the superposition principle and the idea that the inhomogeneity can be viewed as a continuous superposition of delta-like pulses in time, each of which for itself would merely adjust the initial data ‘instantaneously’.

As usual, this motivational insight is one thing, and a rigorous proof is then supplied, which does not require the theory of distributions.

- We are about to establish a (rather uninspiring) mean value property as a tool to prove a (fundamentally) useful maximum principle.

- In tune with the book, we will postpone discussion of the heat equation in bounded domains, because it requires Fourier series (and eigenvalue problems), which we will study later; thereafter, lot of insight into the heat equation in bounded domains is gained easily. Likewise, a convincing motivation for the (presently ‘out of the blue’) fundamental solution can then be supplied.