Distinguishing Relative Minima; Caratheodory's Approach and the Hamilton Formalism

We'll assume $L \in C^3$ throughout this chapter.

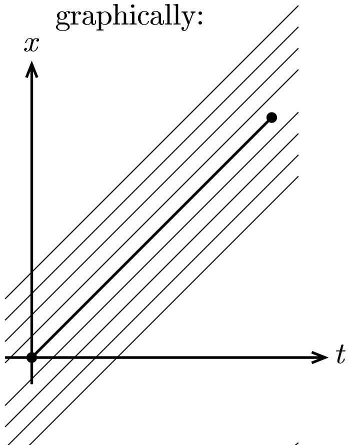
(1) Paradigm: In order to show that the solution $x_*(t) = t$ of the EL equation of the variational problem $I[x] := \int_0^1 \dot{x}^2(t)\,dt$, $x(0) = 0$, $x(1) = 1$ is indeed a minimizer, we can argue as follows (let's ignore for a moment that the convexity of $I[\cdot]$ provides a much easier argument, because the method discussed here is meant to be useful in more generality):

$$I[x] = \int_0^1 \dot{x}(t)^2\,dt = \int_0^1 (\dot{x}(t)-1)^2\,dt \quad + \quad \int_0^1 (2\dot{x}(t)-1)\,dt$$

$$= \int_0^1 (\dot{x}(t)-1)^2\,dt \quad + \quad \Big[2x(t) - t\Big]_0^1$$

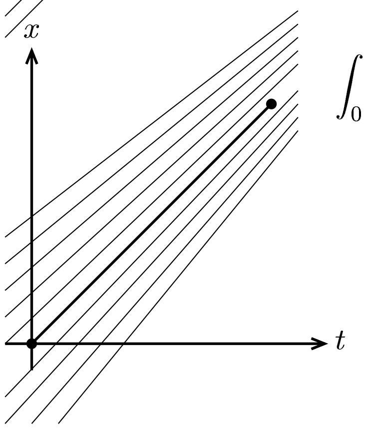
Slope 1 "cost free" extra charge for other slopes + path independent fixed cost 1 $(= I[x_*])$

graphically:



In this figure, the thin lines designate the "cost free" directions; they themselves are solutions to the EL equation.

We can do it differently, but still within the same paradigm:



$$\int_0^1 \dot{x}(t)^2\,dt = \int_0^1 \left(\dot{x}(t) - \frac{2-x(t)}{2-t}\right)^2 dt - \int_0^1 \frac{d}{dt}\left(\frac{(2-x(t))^2}{2-t}\right) dt$$

"cost free slope" depends on position + path independent "flat fee" 1

In this example, the extremals $x(t;a) = a + (1-\frac{a}{2})t$ define the cost free direction. Eliminating a from this equation and its t-derivative leads indeed to $\dot{x} = \frac{2-x}{2-t}$ as the ODE for this family of extremals. Here as in the previous example, the cost free directions define minimals, because the other contribution is a flat fee that does not depend on the path.

(2) This method is surprisingly general:

Let $(t,x) \mapsto \psi(t,x)$, $\mathbb{R}\times\mathbb{R}^n \supset \mathcal{G} \to \mathbb{R}^n$ be a direction field defined in a neighbourhood $\mathcal{G}$ of an extremal x_*. Note that in this context, the word neighborhood does not refer to function space, but to the space $\mathbb{R}\times\mathbb{R}^n$, which contains the graph of the function $x(\cdot)$. The integral curves of this direction field (i.e., the solutions to the ODE $\dot{x} = \psi(t,x)$) should be extremals themselves, and x_* should be among them. It is surely necessary that the integral curves be extremals, if the directions $\psi(t,x)$ are "cost free" and therefore minimal.

Our goal is therefore to determine a vector field $\psi(t,x)$, and a scalar valued function $S(t,x)$, such that we can write for arbitrary curves $x(\cdot)$:

$$L(t,x(t),\dot{x}(t)) = \tilde{L}(t,x(t),\dot{x}(t)) + \tfrac{d}{dt}S(t,x(t)) \quad , \quad \text{i.e.:}$$
$$L(t,x,\dot{x}) = \tilde{L}(t,x,\dot{x}) + S_t(t,x) + \dot{x}S_x(t,x) \text{ for all } (t,x,\dot{x})\,,$$

where for each (t, x), the function $\tilde{L}(t, x, \cdot)$ should have an absolute minimum at $\psi(t, x)$, and the value of this absolute minimum should be 0. This requires, from calculus:

$$\begin{aligned} &L_{\dot{x}}(t, x, \psi(t, x)) = S_x(t, x) \\ &L(t, x, \psi(t, x)) - \psi(t, x) L_{\dot{x}}(t, x, \psi(t, x)) = S_t(t, x) \end{aligned} \tag{2.1}$$

In order that we can find a potential function (antiderivative) S, satisfying the conditions $S_{x_i} = a_i$, $S_t = b$, it is necessary that the integrability conditions $\partial_{x_j} a_i = \partial_{x_i} a_j$ and $\partial_{x_j} b = \partial_t a_j$ hold. These conditions are also sufficient, provided the domain of the functions a_i and b is simply connected, and this topological constraint on $\mathcal{G}$ can easily be achieved in a neighborhood of the extremal x_*, which is where we want to construct the functions ψ and S. In the scalar valued case $n = 1$, only the integrability condition $\partial_x b = \partial_t a$ occurs, and it is easy to check that this condition is equivalent to the condition that the integral curves of ψ (i.e., the solutions to $\dot{x}(t) = \psi(t, x(t))$) be extremals:

When we prove this, we must be careful to distinguish the partial derivatives $\partial/\partial t$, which apply to 3-variable functions in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, with the variables $(t, x, \dot{x})$, from partial derivatives $d/\partial t$ that apply to 2-variable functions in $\mathbb{R} \times \mathbb{R}^n$, with the variables (t, x). These latter arise form plugging $\dot{x} = \psi(t, x)$ into the 3-variable functions. Furthermore, we have the total derivative d/dt, which applies to functions of t alone, once a particular curve $t \mapsto x(t)$ has been substituted for the variable x. To clarify this distinction, we mark all those terms with arrows that get hit by the corresponding time derivative. A similar convention will distinguish partial derivatives $\partial/\partial x_i$ from $d/\partial x_i$.

Written in components, the integrability condition $\partial_t a_i = \partial_{x_i} b$ reads therefore:

$$\frac{d}{\partial t} L_{\dot{x}_i}(t, x, \psi(t, x)) = \frac{d}{\partial x_i} \left(L(t, x, \psi(t, x)) - \sum_j \psi_j(t, x) L_{\dot{x}_j}(t, x, \psi(t, x)) \right)$$

$$\frac{d}{\partial t} L_{\dot{x}_i}(t, x, \psi(t, x)) + \sum_j \psi_j(t, x) \frac{d}{\partial x_i} L_{\dot{x}_j}(t, x, \psi(t, x)) =$$

$$= \frac{d}{\partial x_i} L(t, x, \psi(t, x)) - \sum_j \left(\frac{\partial}{\partial x_i} \psi_j(t, x) \right) L_{\dot{x}_j}(t, x, \psi(t, x))$$

which is an equation for functions of the variables (t, x). Evaluating this equation on the graph of an integral curve $t \mapsto x(t)$ of the vector field ψ ($\dot{x}(t) = \psi(t, x(t))$), this reduces to

$$\frac{d}{dt} L_{\dot{x}_i}(t, x(t), \psi(t, x(t))) = L_{x_i}(t, x(t), \psi(t, x(t))) \ ,$$

which is exactly the condition that integral curves satisfy the EL equation. Here, in order to identify the left hand side $dL_{\dot{x}_i}/\partial t + \sum \psi_j dL_{\dot{x}_j}/\partial x_i$ with $dL_{\dot{x}_i}/dt$, we have used the other integrability condition $dL_{\dot{x}_j}/\partial x_i = dL_{\dot{x}_i}/\partial x_j$.

So for $n = 1$ all we need to do is to cover a neighborhood of an extremal segment x_* with further extremals in such a way that through each point exactly one extremal segment passes; then the slope of the extremal passing through (t, x) defines $\psi(t, x)$. For $n > 1$, we cannot cover the neighborhood with extremals in any arbitrary way, but our choice of extremals has to satisfy further integrability conditions. We will see later that this is indeed possible under general hypotheses. In any case, we have almost proved the following Lemma:

(3) Lemma: If it is possible to define a vector field $\psi : (t, x) \mapsto \psi(t, x) \in \mathbb{R}^n$ in some neighborhood of an extremal $t \mapsto x_*(t)$, $[t_0, t_1] \to \mathbb{R}^n$ that $\dot{x}_*(t) = \psi(t, x_*(t))$, and that for

$$a_i(t, x) := L_{\dot{x}_i}(t, x, \psi(t, x)) \tag{3.1}$$

$$b(t,x) := L(t,x,\psi(t,x)) - \sum_j \psi_j(t,x) L_{\dot{x}_j}(t,x,\psi(t,x)) \tag{3.2}$$

the integrability conditions

$$\partial_{x_j} a_i = \partial_{x_i} a_j \tag{3.3}$$

$$\partial_{x_j} b = \partial_t a_j \tag{3.4}$$

are satisfied, then one can write, in terms of an antiderivative S (i.e., $\partial_{x_j} S = a_j$, $\partial_t S = b$):

$$\int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))\,dt = \int_{t_0}^{t_1} \mathfrak{E}(t,x(t),\psi(t,x(t)),\dot{x}(t))\,dt + \Big[S(t,x(t))\Big]_{t_0}^{t_1} ,$$

where

$$\begin{aligned}\mathfrak{E}(t,X,\psi,\dot{X}) &:= L(t,X,\dot{X}) - L(t,X,\psi) - (\dot{X}-\psi)L_{\dot{x}}(t,X,\psi) \\ &= (\dot{X}-\psi)^2 \int_0^1 L_{\dot{x}\dot{x}}(t,X,(1-s)\psi + s\dot{X})\,ds \,.\end{aligned}$$

Proof:

$$\int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))\,dt = \int_{t_0}^{t_1} \underbrace{(L(t,x,\dot{x}) - a\dot{x} - b)}_{\mathfrak{E}(\ldots)}\,dt + \int_{t_0}^{t_1} \underbrace{(a\dot{x}+b)}_{\frac{d}{dt}S(t,x(t))}\,dt \,.$$

Remarks:

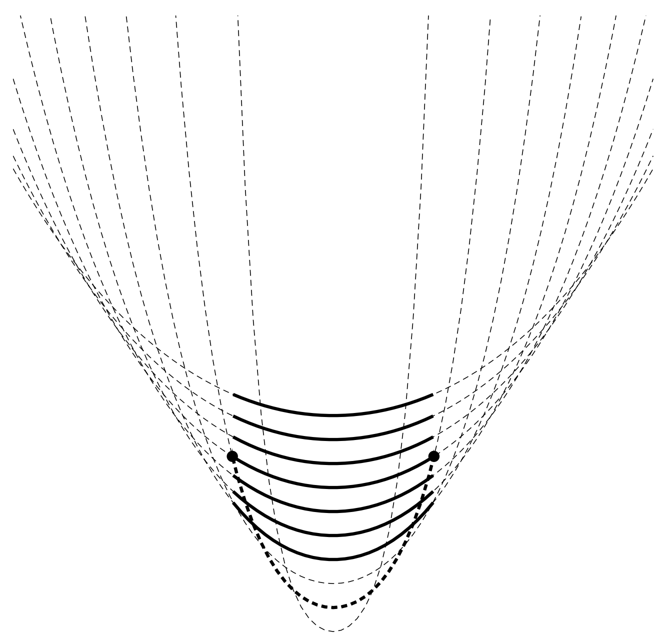
(1) The path independent integral $S = \int (L + (\dot{x}-\psi)L_{\dot{x}}(t,x,\psi))\,dt$ is called *Hilbert's invariant integral.*

(2) The lemma is useful because we will be able to show the existence of ψ under hypotheses that can be checked easily.

(3) Convexity of L *solely in the variable* $\dot{x}$ guarantees $\mathfrak{E} \geq 0$, and therefore minimality according to paradigm (1), provided the hypotheses of the lemma are fulfilled.

(4) The key obstruction against a construction of ψ occurs, when nearby extremals intersect in the interval $[t_0,t_1]$. However, since the leading order of the difference between nearby extremals obeys the Jaconi equation, conjugate points are the infinitesimal version of this obstruction.

(4) Example: The Catenoid

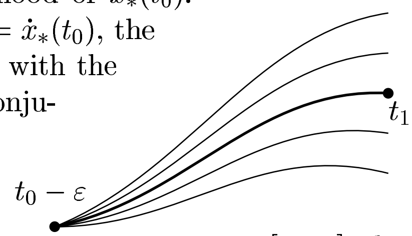


In the figure we sketch the family of extremals we have calculated in the homework. Let us consider the 'flatter' segment between two marked points. The boldface lines describe a natural way to cover a neighborhood of this segment with other extremals. (It is not the only way. Note that horizontal translations $y = E\cosh\frac{x-x_0}{E}$ of extremals are extremals again.)
For the 'steeper' segment, the situation is different. Nearby extremal segments will intersect it. In such points of intersection, $\psi(t,x)$ cannot be defined.

(5) Corollary: Let $n = 1$. If x_* is an extremal on $[t_0, t_1]$, with $L^*_{\dot{x}\dot{x}} > 0$, and if moreover there is no conjugate point of $x_*(t_0)$ in this segment, then x_* is a weak minimal.

If moreover it holds $\mathfrak{E}(t, x, \dot{x}, \dot{X}) \geq 0$ in some neighborhood of the segment, i.e., for all $\dot{X}$, but for $|x - x_*(t)| < \varepsilon$, $|\dot{x} - \dot{x}_*(t)| < \varepsilon$, then x_* is a strong minimal.

Proof:
Consider solutions $t \mapsto x(t; v)$ of the EL equation with the initial conditions $x(t_0) = x_*(t_0)$, $\dot{x}(t_0) = v$, where v ranges over a neighborhood of $\dot{x}_*(t_0)$. In particular we are interested in the dependence on v: At $v = \dot{x}_*(t_0)$, the function $\varphi(t) := \partial x(t; v)/\partial v$ solves the Jacobi equation at x_* with the initial conditions $\varphi(t_0) = 0$, $\dot{\varphi}(t_0) = 1$. Provided there are no conjugate points, the implicit function theorem guarantees that the family of extremals $x(t; v)$ has no other points of intersection than $(t_0, x_*(t_0))$, for v in some neighborhood of $\dot{x}_*(t_0)$. This is not exactly what we want yet; however, if $(t_0, x_*(t_0))$ has no conjugate point on $[t_0, t_1]$, then by continuity, $(t_0 - \varepsilon, x_*(t_0 - \varepsilon))$ has no conjugate point on the segment either, and we use the above argument for this situation. This is how we can foliate (= cover without intersection) a neighborhood of the segment $x_*|_{[t_0,t_1]}$ with extremals.



For a wide (C^0–) neighborhood of x_* we can consider all those curves whose graphs lie in the domain foliated by the extremals. For a narrow (C^1–) neighborhood the subset of those curves that also satisfy $\|\dot{x} - \dot{x}_*\|_\infty < \varepsilon$.

Since

$$I[x] = \int_{t_0}^{t_1} \mathfrak{E}(t, x(t), \psi(t, x(t)), \dot{x}(t))\, dt + I[x_*]$$

it follows that $I[x] \geq I[x_*]$ for all curves, on which $\mathfrak{E} \geq 0$ holds (in every point). This is why x_* is a strong minimal, provided the Weierstrass condition is satisfied. However, if merely $L^*_{\dot{x}\dot{x}} > 0$ is satisfied, then there exists a small neighborhood $\{(t, x, \dot{x}) \mid t \in [t_0, t_1]\,,\ |x - x_*(t)| < \varepsilon\,,\ |\dot{x} - \dot{x}_*(t)| < \varepsilon\} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, on which $L_{\dot{x}\dot{x}} > 0$ still holds. Making ε smaller, if necessary, this domain lies in $\mathcal{G}$. This defines a narrow neighborhood of x_*, and $\mathfrak{E}(t, x(t), \psi(t, x(t)), \dot{x}(t)) \geq 0$ holds in this neighborhood, with equality only for $\dot{x}_* \equiv \dot{x}$.

Our next task is to construct an appropriate extremal field in the case $n > 1$ as well.

(6) Definition: A vector field $\psi(t, x)$ (or the matching family of integral curves) is called a *Mayer field* (for a Lagrange function L), if the integrability conditions (3.3), (3.4) are satisfied. As mentioned, (3.3) and (3.4) imply that the curves form a field of extremals.

(6a) Theorem: An extremal field passing through a fixed initial point is automatically a Mayer field; more precisely: Let $x(\cdot, v)$, for $v \in \mathcal{V} \subset \mathbb{R}^n$, with an open set V, be the family of solutions to the EL equation with the initial conditions $x(t_0; v) = x_0$ fixed, $\dot{x}(t_0; v) = v$. For an interval $]t_0, t_1[$, on which these curves do not intesect (i.e., $x(t; v_1) = x(t; v_2) \wedge t \in]t_0, t_1[\Longrightarrow v_1 = v_2$), a Mayer field is defined by $\psi(t, x(t; v)) := \dot{x}(t; v)$ on $\mathcal{G} := x(]t_0, t_1[, \mathcal{V})$. The injectivity condition for $x(t; \cdot)$ is automatically fulfilled when t_1 is sufficiently close to t_0, and the matrix $\partial x/\partial v$ is invertible there.

Proof: Since $Dx(t; \cdot) = (t - t_0)\mathbf{1} + O(t - t_0)^2$, the injectivity condition for small $|t_1 - t_0|$ follows from the implicit function theorem. As long as injectivity holds, ψ is defined on $\mathcal{G}$ and is C^1. We need to check the integrability condition (3.3). Letting

$$p_i(t; v) := a_i(t, x(t; v)) = L_{\dot{x}_i}(t, x(t; v), \dot{x}(t; v))$$

it reads $\partial a_i/\partial x_j = \partial a_j/\partial x_i$. Since the matrix $\partial x/\partial v$ is invertible in $\mathcal{G}$, the condition is equivalent to

$$\sum_{i,j} \frac{\partial a_i}{\partial x_j}\,\frac{\partial x_j}{\partial v_k}\,\frac{\partial x_i}{\partial v_r} = \sum_{i,j} \frac{\partial a_j}{\partial x_i}\,\frac{\partial x_j}{\partial v_k}\,\frac{\partial x_i}{\partial v_r}\,.$$

Using the chain rule for $p(t;v) = a(t, x(t;v))$, this amounts to

$$\sum_i \left(\frac{\partial p_i}{\partial v_k}\,\frac{\partial x_i}{\partial v_r} - \frac{\partial p_i}{\partial v_r}\,\frac{\partial x_i}{\partial v_k} \right) = 0\,. \tag{6.1}$$

The expression on the left hand side is called Lagrange bracket of the curve family $x(t;\cdot)$. If the curve family consists of *extremals*, the Lagrange bracket depends on v only, but not on t, as we will see in a moment. This allows us to test condition (6.1) for $t = t_0$ where it holds trivially.

Let us show that the Lagrange bracket is indeed constant in time, for a family of extremals:

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_i \left(\frac{\partial p_i}{\partial v_k}\,\frac{\partial x_i}{\partial v_r} - \frac{\partial p_i}{\partial v_r}\,\frac{\partial x_i}{\partial v_k} \right) = \sum_i \left(\left(\frac{\partial}{\partial v_k}\frac{\partial p_i}{\partial t}\right)\frac{\partial x_i}{\partial v_r} + \frac{\partial p_i}{\partial v_k}\,\frac{\partial \dot x_i}{\partial v_r} - (k \leftrightarrow r) \right)\\
&= \sum_i \left(\left(\frac{\partial}{\partial v_k} L_{x_i}\right)\frac{\partial x_i}{\partial v_r} + \left(\frac{\partial}{\partial v_k} L_{\dot x_i}\right)\frac{\partial \dot x_i}{\partial v_r} - (k \leftrightarrow r) \right)\\
&= \sum_{i,s} \left(L_{x_i x_s}\frac{\partial x_s}{\partial v_k}\,\frac{\partial x_i}{\partial v_r} + L_{x_i \dot x_s}\frac{\partial \dot x_s}{\partial v_k}\,\frac{\partial x_i}{\partial v_r} + L_{\dot x_i x_s}\frac{\partial x_s}{\partial v_k}\,\frac{\partial \dot x_i}{\partial v_r} + L_{\dot x_i \dot x_s}\frac{\partial \dot x_s}{\partial v_k}\,\frac{\partial \dot x_i}{\partial v_r} - (k \leftrightarrow r) \right)\\
&= 0
\end{aligned}$$

Here $\partial/\partial t$ is actually the total time derivative, written with curly ∂ only because of the presence of the other variable v. This proves the theorem, and we have therefore
(6b) Theorem: Corollary (5) is also valid without the hypothesis $n = 1$, the argument being the same.

This proves the sufficient conditions listed on the second survey sheet on minimality notions. Refer to the homework for the necessary conditions not discussed so far.

(7) Theorem (with Def.): Under the hypothesis $L_{\dot x \dot x} > 0$ (i.e., if this matrix is positive definite), the mapping $\dot x \mapsto L_{\dot x}(t, x, \dot x) =: p$ is globally one-to-one, and the EL equation $\frac{d}{dt}L_{\dot x}(t, x(t), \dot x(t)) = L(t, x(t), \dot x(t))$ can be written as a system of *Hamilton equations*

$$\begin{aligned}
\tfrac{d}{dt}p(t) &= -H_x(t, x(t), p(t))\\
\tfrac{d}{dt}x(t) &= H_p(t, x(t), p(t))
\end{aligned} \tag{7.1}$$

where the *Hamilton function* H is defined by

$$\begin{aligned}
&H(t, x, p) = \dot x L_{\dot x}(t, x, \dot x) - L(t, x, \dot x)\\
&p = L_{\dot x}(t, x, \dot x)
\end{aligned} \tag{7.2}$$

This transformation is called *Legendre transformation.*

Proof: We discussed the injectivity of the Legendre transformation $p \leftrightarrow \dot x$ at the end of Chapter 2 in the lecture. The implicit function theorem guarantees that the transformation and its inverse are C^1, provided $L \in C^2$. Let us study this transformation *kinematically* (i.e., with the three variables $(t, x, \dot x)$ treated as independent variables. If we differentiate

$$H(t, x, L_{\dot x}(t, x, \dot x)) = \dot x L_{\dot x}(t, x, \dot x) - L(t, x, \dot x)$$

with respect to $\dot x_i$ or x_i respectively, we get:

$$\begin{aligned}
\textstyle\sum_j H_{p_j} L_{\dot x_j \dot x_i} &= L_{\dot x_i} + \textstyle\sum_j \dot x_j L_{\dot x_j \dot x_i} - L_{\dot x_i}\\
H_{x_i} + \textstyle\sum_j H_{p_j} L_{\dot x_j x_i} &= \textstyle\sum_j \dot x_j L_{\dot x_j x_i} - L_{\dot x_i}
\end{aligned} \quad \text{and hence} \quad \begin{aligned} H_p &= \dot x\\ H_x &= -L_x \end{aligned}$$

From this it follows (now *dynamically*, i.e., with actual solutions $x(t)$ plugged in):

$$\frac{d}{dt}p(t) \stackrel{\text{Def. } p}{=} \frac{d}{dt}L_{\dot{x}}(t,x(t),\dot{x}(t)) \stackrel{\text{EL eqn}}{=} L_x(t,x(t),\dot{x}(t)) \stackrel{\text{coord trf}}{=} -H_x(t,x(t),p(t))$$
$$\frac{d}{dt}x(t) = \dot{x}(t) \stackrel{\text{coord trf}}{=} H_p(t,x(t),p(t))$$

By the way, it can be shown in the same way that $L_{\dot{x}\dot{x}}$ and H_{pp} are each other's inverse matrix, and in particular H_{pp} is positive definite.

Remark: The function H is a familiar friend: In the case of $L_t = 0$, we had written a consequence of the EL equations as $\frac{d}{dt}H = 0$. In mechanics, H ist the energy, $\dot{x}$ the velocity and p the momentum.

In the Hamiltonian framework, equations (3.1), called the *fundamental equations of the calculus of variations*, can be written as

$$\begin{aligned}\partial_{x_i}S &= L_{\dot{x}_i}(t,x,\psi(t,x)) = p_i(t,x)\\ \partial_t S &= L(t,x,\psi(t,x)) - \psi(t,x)L_{\dot{x}}(t,x,\psi(t,x)) = -H(t,x,p(t,x))\end{aligned}$$

and they decouple:

$$S_t(t,x) + H(t,x,S_x(t,x)) = 0 \tag{7.3}$$
$$\psi(t,x) = H_p(t,x,S_x(t,x)) \tag{7.4}$$

We therefore have the

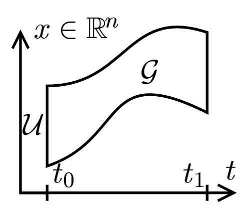
(8) Corollary: The hypotheses of Lemma (3), i.e., the existence of a Mayer field ψ in some neighborhood of an extremal, is equivalent to the solvability of equation (7.3), where H is the Hamilton function associated to the Lagrange function L.

(8a) Definition: The partial differential equation (7.3) is called the *Hamilton Jacobi equation.*

Its solvability is equaivalent to the existence of a Mayer field, and inasmuch as needed, we have proved it already. Now there is a more general principle behind the relation between the Hamilton equations (7.1) and the Hamilton Jacobi equation, and we will discuss this connection now. Partial differential equations of first order (and also a selection of higher order PDEs like the wave equation) can be reduced to ordinary differential equations by means of the *Method of Characteristics.*

(9) Theorem: Given $H \in C^2$, $H_{pp} > 0$ (i.e., positive definite), $\mathcal{U}$ a bounded domain in $\mathbb{R}^n$, and a function $g \in C^3(\bar{\mathcal{U}} \to \mathbb{R})$ (meant to describe initial conditions), a solution $S \in C^3$ of the so-called *Cauchy problem* for the Hamilton Jacobi equation

$$\begin{aligned}&S_t + H(t,x,S_x) = 0 &&\text{for } (t,x) \in \mathcal{G}; \quad \mathcal{G} \text{ chosen appropriately}\\ &S(t_0,x) = g(x) &&\text{for } x \in \mathcal{U}; \quad \mathcal{U} \text{ given}\end{aligned} \tag{9.1}$$



can be constructed as follows: Consider the initial value problem for the system of ODEs

$$\begin{aligned}\dot{x}(t) &= H_y(t,x(t),y(t)) & x(t_0) &=: u \in \mathcal{U}\\ \dot{y}(t) &= -H_x(t,x(t),y(t)) & y(t_0) &= \nabla g(u)\\ \dot{z}(t) &= -H(t,x(t),y(t)) + y(t)H_y(t,x(t),y(t)) & z(t_0) &= g(u)\end{aligned} \tag{9.2}$$

If the map $(t,u) \mapsto (t,x)$, $[t_0,t_1] \times \mathcal{U} \to \mathcal{G}$ (the projection of the flow map on the x components) is bijective, then the Cauchy problem has a unique solution in $\mathcal{G}$; however if this map is not injective, the problem does not have a solution in $\mathcal{G}$. In the former case, for given $(t, x) \in \mathcal{G}$, an initial value u can be determined uniquely so that the solution passes through (t, x). Then, in terms of the solution to this initial value, one can define $z(t) =: S(t, x)$.

Preliminary comment on the proof: The method of characteristics is the following: In the domain in which we want to find solutions of the PDE, we look for certain curves along which the PDE reduces to an ODE. These curves are called characteristic curves. Such curves only exist for certain PDEs. In a very simple example, for the PDE $u_x(t, x) = 2u_t(t, x)$, the curves $x = a - t/2$ are characteristic, because the substitution $w(t) := u(t, a - t/2)$ reduces the PDE to the ODE $w'(t) = 0$. This is the method that we are going to use for the proof. The Hamilton equations are exactly the ODEs that determine characteristic curves.

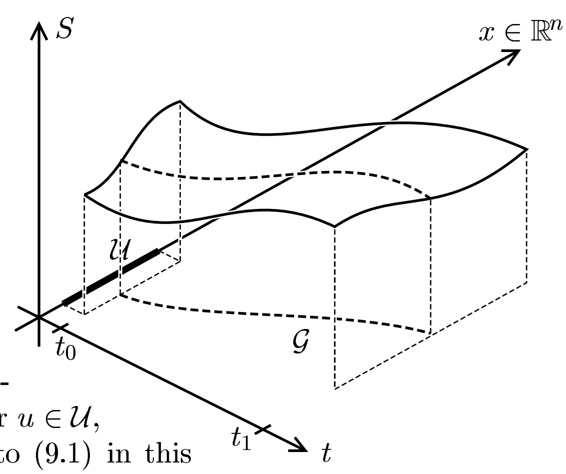
Proof: Let us first assume we do have a solution $S(t, x)$ of (9.1), and we define on any curve of the form $(t, x(t))$ the function $z(t) := S(t, x(t))$ (which then obviously depends on the curve we chose). Our motivation is that we now want to choose the curve in such a way that the PDE (7.3) reduces to an ODE for z. We find

$$\dot{z}(t) = S_t(t, x(t)) + \textstyle\sum_j S_{x_j}(t, x(t))\dot{x}_j(t) = -H(t, x(t), S_x(t, x(t))) + \sum_j S_{x_j}(t, x(t))\dot{x}_j(t)$$

Thus far, we have not succeeded yet to eliminate S: The value of S_x *on the curve* cannot be determined from S *on the curve alone*. Therefore we also define $y(t) := S_x(t, x(t))$ and attempt to choose our curve in such a way that an ODE involving z and y results. So we find, using the derivative of (7.3), namely $S_{tx_k} + H_{x_k}(t, x, S_x(t, x)) + \sum_j H_{y_j}(t, x, S_x(t, x))S_{x_j x_k}(t, x) = 0$), that

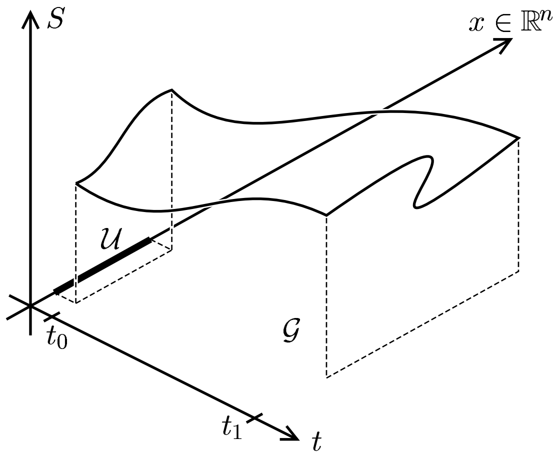
$$\begin{aligned}\dot{y}_k(t) &= S_{x_k t}(t, x(t)) + \textstyle\sum_j S_{x_k x_j}(t, x(t))\dot{x}_j(t) \\ &= -H_{x_k}(t, x(t), y(t)) - \textstyle\sum_j S_{x_k x_j}(t, x(t))\Big(H_{y_j}(t, x(t), y(t)) - \dot{x}_j(t)\Big)\end{aligned}$$

Therefore, in order that we do not need to enter further derivatives of S into our system of ODEs, we require that the big parenthesis term vanishes, and this requirement gives us an ODE for the characteristic curves, namely exactly (9.2).



Wrapping the argument up, we consider a solution S of (9.1) on a curve segment $x(t)$ in the domain of S, definied by the Hamilton equations $\dot{x} = -H_y$, $\dot{y} = -H_x$. On such a segment, the function $z(t) := S(t, x(t))$ satisfies the ODE $\dot{z} = -H + yH_y$, which is obtained by substituiton from the equation $\dot{z} = -H + S_x\dot{x}$.

If we prescribe values for S on $\{t_0\} \times \mathcal{U}$, then the values of S_x are defined there as well, by taking the derivatives with respect to x; this gives us the initial conditions in (9.2). So if $\mathcal{G}$ is the domain determined by solving the Hamilton equations (9.2) for $u \in \mathcal{U}$, then we have shown that any possible solution S to (9.1) in this domain is uniquely determined by the $\dot{z}$ equation among (9.2).



Conversely, however, it is not certain that any solution of (9.2) gives rise to a solution of (9.1). For it could happen that solutions (t, x, y, z) with different initial values nevertheless develop the same coordinates (t, x). In the figure this can be seen as the phenomenon that the surface generated from the solution curves to (9.2) is not the graph of a function but "folds back". In this case, equation (9.1) does not have a solution in $\mathcal{G}$.

However, if we make the injectivity assumption of the theorem and define $S(\hat{t}, \hat{x})$ for $(\hat{t}, \hat{x}) \in \mathcal{G}$ as the value $z(\hat{t})$ of the *unique* solution of (9.2) that satisfies $x(\hat{t}) = \hat{x}$, then we are assuming the existence of functions $x(t;u)$, $y(t;u)$, $z(t;u)$, that satisfy the IVP (9.2) and also the equation

$$z(t;u) = S(t, x(t;u)) \tag{9.3}$$

for some function S defined in $\mathcal{G}$. Since $\nabla g \in C^2$, the dependence $x \leftrightarrow u$ is also C^2, and we conclude $S \in C^2$. We want to show that

$$S_t(t, x(t;u)) = -H(t, x(t;u), y(t;u)) \ , \quad S_x(t, x(t;u)) = y(t;u)$$

hold. Then the validity of the PDE (9.1) is immediate, and also $S \in C^3$ since $x, y \in C^2$.

To this end we obviously must make use of the initial conditions in (9.2), because without them the claim wouldn't even be valid for $t = t_0$. So take the time derivative of (9.3) and obtain, by means of (9.2),

$$-H + yH_y = S_t + S_x H_y$$

or

$$S_t(t, x(t;u)) + H(t, x(t;u), y(t;u)) = H_y(t, x(t;u), y(t;u))\Big(y(t;u) - S_x(t, x(t;u))\Big) \tag{9.4}$$

It suffices to show $S_x = y$, then $S_t = -H$ follows automatically. We also use the relation just obtained to eliminate S_t and the time derivative in S_{xt}. From (9.4), we get by means of differentiation with respect to u:

$$\sum_j \Big(S_{tx_j} + H_{x_j}\Big)\frac{\partial x_j}{\partial u_k} + H_{y_j}\frac{\partial y_j}{\partial u_k} = \sum_j \left\{ \frac{\partial H_{y_j}(\ldots)}{\partial u_k}\Big(y_j - S_{x_j}\Big) + H_{y_j}\Big(\frac{\partial y_j}{\partial u_k} - \sum_i S_{x_j x_i}\frac{\partial x_i}{\partial u_k}\Big)\right\} \tag{9.5}$$

and therefore

$$\sum_j \Big(S_{tx_j} + H_{x_j} + \sum_i H_{y_i} S_{x_i x_j}\Big)\frac{\partial x_j}{\partial u_k} = \sum_j \frac{\partial H_{y_j}(\ldots)}{\partial u_k}\Big(y_j - S_{x_j}\Big) \tag{9.6}$$

Now $S_x(t, x(t;u)) = y(t;u)$ holds at least for $t = t_0$, due to the initial conditions. So all we need to show is the time derivative of this equation. Now it holds

$$\frac{\partial}{\partial t}\Big(S_x(t, x(t;u)) - y(t;u)\Big) = S_{xt} + S_{xx}(t, x(t;u))H_y + H_x$$

and therefore form (9.6)

$$\frac{\partial x}{\partial u} \cdot \frac{\partial}{\partial t}(S_x - y) = -\frac{\partial H_y}{\partial u} \cdot (S_x - y)$$

By the uniqueness theorem (and the invertibility of the matrix $\partial x / \partial u$ it follows that $S_x - y = 0$ holds for all times.