Homework
UTK – M534 – Calculus of Variations
Spring 2014, Jochen Denzler, MWF 11:15–12:05

1. The Brachystochrone Problem: Among ‘all’ planar curves $(x, y(x))$, $x \in [0, x_0]$, connecting $(0, 0)$ to (x_0, y_0), find (the) one that minimizes the time a point of mass would take to slide from $(0, 0)$ to (x_0, y_0), beginning at rest and moving solely under the influence of gravity, in the absence of friction.

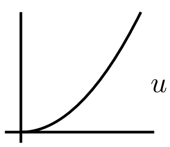
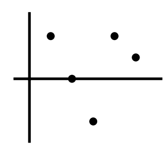
Let’s view this as a modelling problem first: Explain how this problem translates to the following mathematical formulation:

$$\min\left\{ I[y] := \int_0^{x_0} \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, dx \;\middle|\; y \in C^0[0, x_0]\,,\ y(0) = 0\,,\ y(x_0) = y_0\,,\ I[y] \text{ defined} \right\}$$

Suggest a precise class of admissible functions $y(\cdot)$ for which $I[y]$ is guaranteed to be defined.

2. Informal analogy: Functions as ‘infinite dimensional vectors’: In comparing Calculus of Variations with Multivariable Calculus, the analog of a function $u : x \mapsto u(x)$ is a vector $\vec{u} = (u_1, \ldots, u_n)$. The vector $\vec{u}$ should not be represented as an arrow in n-dimensional space, but rather like the graph of a function $i \mapsto u_i$ that is defined only for $i \in \{1, \ldots, n\}$.

The calc’s are a bit messy, and you may want to use technology, even though it can be done without. This is a sample of the Rayleigh-Ritz and (relatedly) of the finite element method. But for the moment, the focus is that you absorb the idea from the boldfaced title of the problem.


$u : x \mapsto x^2$ compare with $\vec{u} = (1, 0, -1, 1, \frac{1}{2})$


Let us compare the problem

$$\min\left\{ I[u] := \int_0^1 (u'(x)^2 - u(x)^2 + 2u(x))\, dx \;\middle|\; u \in \text{piecewise}C^1[0, 1]\,,\ u(0) = 0 = u(1) \right\}$$

with an appropriate minimum problem in $\mathbb{R}^3$: namely
$\min\{I_3(\vec{u}) \mid \vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3\}$.

Task 1 is to find an appropriate function I_3 that is a fair analog $I[\cdot]$. Here is how: Given $\vec{u} = (u_1, u_2, u_3)$, let $\hat{u}$ be the piecewise linear function satisfying $\hat{u}(0) = 0 = \hat{u}(1)$ and $\hat{u}(\frac{1}{4}) = u_1$, $\hat{u}(\frac{2}{4}) = u_2$, $\hat{u}(\frac{3}{4}) = u_3$, linear on each of the segments $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$. Then we let $I_3(\vec{u}) := I[\hat{u}]$. Calculate a formula for I_3 and use it to set up the equations $DI(\vec{u}) = 0$. Solve these equations. (They have a unique solution $\vec{u}_*$.) Next show that the Hessian $D^2I(\vec{u}_*)$ is indeed positive definite.

Task 2 is to set up a corresponding equation for the original problem $\min I[u]$. We’ll soon learn how to do this. Until then, I just give you the answer: it is the ODE $u'' + u = 1$, together with the boundary conditions $u(0) = 0 = u(1)$. Now solve this ODE (BVP). (It has exactly one solution u_*.) For now, don’t worry about any possible meaning of a ‘Hessian’ $D^2I[u_*]$ or what positive definiteness of such a ‘Hessian’ may mean.

Task 3 is a simple comparison: Plot $\vec{u}_*$, $\hat{u}_*$ and u_* in one coordinate system.

Note: In a thorough theory, one allows any u such that $u' \in L^2$, rather than requiring piecewise C^1. We are avoiding advanced-calculus technicalities here by being content with piecewise C^1 candidate functions u.

3. A simple minimization problem without a solution: Show that the problem

$$\min \left\{ I[u] := \int_0^1 u(x)^2 (1-u(x))^2 \, dx \;\middle|\; u \in C^0[0,1]\,,\; u(0)=0\,,\; u(1)=1 \right\}$$

has no solution because the infimum is not taken on. What number is the infimum? Suggest a larger, still sensible, class of admissible functions within which the minimum is taken on. How many minimizers do there exist in this larger class?

Those who have heard about Banach spaces should think about whether the larger class of admissible functions being suggested is a Banach space. (Strictly, speaking: a closed affine subspace of a Banach space, because the boundary condition $u(1) = 1$ kills any vector space structure.)

4. Saddle Point at Infinity

This example is taken from *I. Rosenholtz, L. Smylie: "The only Critical Point in Town" Test*, Mathematics Magazine **58**(1985), 149–150.

Show that the function

$$I : (x,y) \mapsto y^2 + 3(y + e^x - 1)^2 + 2(y + e^x - 1)^3\,,\; \mathbb{R}^2 \to \mathbb{R}$$

has exactly one critical point. ('Critical point' means: a point where the derivative vanishes.) Also calculate the Hessian and show that this critical point is a relative minimum. Furthermore notice that the function I is unbounded below. In particular the relative minimum is not an absolute minimum.

The moral of this is: the only solution to the equation $DI(u) = 0$ need not be an absolute minimum, even if it is a relative minimum. This phenomenon can already occur if $u = (x,y) \in \mathbb{R}^2$, so we cannot expect a better situation in Calculus of Variations, where u lies in an infinite dimensional space. Making the extra assumption that I should be bounded below does not help the situation, because then $u \mapsto \exp I(u)$ still serves as a counterexample.

Can you guess why the title of this problem is "saddle point at infinity"? More precisely, can you supply a motivation how this example was constructed?

In single-variable calculus, this phenomenon cannot happen; indeed show: Suppose for a function $I \in C^1(\mathbb{R})$ there is *exactly one* solution u_0 to the equation $I'(u) = 0$, and that u_0 is a relative minimum. Then u_0 is an absolute minimum.

5. Sailing down the river, against headwind: It is possible to use headwind to sail against the wind. The optimal strategy steers a zigzag course with a 45° against the wind, while using an appropriate positioning of the sail. How this works is an application of appropriate decompositions of the force vector, the (quite elementary) details of which are of no concern here. If you are sailing down a river, against the wind, you may also want to use the current (which is largest in the mid of the river) in your favor.

Consider the problem to minimize (if possible)

$$I[u] := \int_{-1}^{1} \left((1 - u'(x)^2)^2 + u(x)^2 \right) \, dx$$

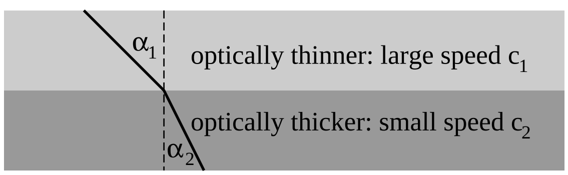
Construct a sequence of piecewise C^1 functions u_n such that $I[u_n] \to 0$. In contrast show also that for every piecewise C^1 function u, it holds $I[u] > 0$. So a minimum of I again fails to exist.

Explain the dilemma in mathematical terms as well as, qualitatively, in the analogy of the sailing problem.

***6.** *This may be useful for present and aspiring Math graduate students, but is not required for the course. Prereq: Know implicit function theorem.* Work out the details of the proof of the Lagrange multiplier theorem in $\mathbb{R}^n$, using the implicit function theorem.

***7.** *This is just a refresher problem in case somebody finds it helpful.* Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(0,0) = 0$ and $f(x,y) = x^2y/(x^2+y^2)$ otherwise, is continuous at the origin and has directional derivatives in each direction $(\cos\varphi, \sin\varphi)$ there, but that the graph does not have a tangential plane (the function is not differentiable) there. Calculate the partial derivatives $\partial_1 f(x,y)$ and $\partial_2 f(x,y)$. Show that indeed they are not continuous at the origin.

8. **Johann Bernoulli's Solution to the Brachystochrone Problem:** The solution alluded to here is outlined in the book "The parsimonious universe" by St. Hildebrandt and A. Tromba, chapter 3. It consists of an optical re-interpretation of the same mathematical model. Namely in an optical medium of varying optical density (i.e., varying speed of light), a light ray is bent. Fermat's principle states that the path of light between two points will be the one that takes the shortest time. To translate the brachystochrone problem (Hwk #1) into a light travel problem, assume that the light speed at height y is $\sqrt{y}$. In a discontinuous medium, Fermat's principle translates into Snell's law of refraction, namely



$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{c_1}{c_2}$$

as can be seen by solving a Calc'1 style minimum problem. So Bernoulli first considered the medium as consisting of many, but finitely many, thin slices in each of which the light speed is constant, gave a condition for the light path from Snell's law, and then calculated the limit where the number of slices goes to infinity with their thickness shrinking to 0 appropriately.

Obtain the ODE for the brachystochrone according to this method. (Never mind solving it.) Also obtain the ODE as Euler–Lagrange equation directly the "modern" way. (Never mind justifications for the exchange of $\frac{d}{d\varepsilon}$ with $\int \ldots dx$.)

9. **The Catenoid, I:**
Find (directly) the Euler-Lagrange equation for the problem of a rotational surface of minimal area,

$$\min\left\{ I[r] := 2\pi \int_{-x_1}^{x_1} r(x)\sqrt{1+r'(x)^2}\,dx \;\middle|\; r \in C^1([-x_1,x_1] \to \mathbb{R}^+),\ r(\pm x_1) = r_1 \right\}.$$

(We'll solve it later).

10. **Erdmann's Corner Condition**
Let $L \in C^1(]t_0,t_1[\times \mathcal{G} \times \mathbb{R}^n \to \mathbb{R})$ with $\mathcal{G} \subset \mathbb{R}^n$ open. Assume y^* is piecewise C^1 where a corner may possibly occur at a certain $\hat{t} \in]t_0,t_1[$. If such a y^* is a weak mimimum for the variational problem

$$I[y] := \int_{t_0}^{t_1} L(t, y(t), \dot{y}(t))\,dt\,, \quad y(t_0) = y_0\,,\ y(t_1) = y_1\,,$$

then the usual Euler equation holds for $t \neq \hat{t}$ (explain why). Show that at $t = \hat{t}$, the corner condition $p^*(t-) = p^*(t+)$ must be verified, where

$$p(t) := L_{\dot{y}}(t, y(t), \dot{y}(t)) \; .$$

In other words, even if $\dot{y}^*$ should have a jump discontinuity at $\hat{t}$, nevertheless p^* must be continuous there.

11. Corners may occur
Find (by direct inspection) all solutions to the variational problem

$$\min \left\{ \int_0^3 \dot{y}^2(t)(1 - \dot{y}(t))^2 \, dt \;\middle|\; y \text{ piecewise } C^1 \, , \; y(0) = 0 \, , \; y(3) = 1 \right\}$$

and confirm that Erdmann's corner condition is verified at corners.

Also find the Euler equation of the variational problem and show that the only C^1 solution (with NO corners) on $[0, 3]$ subject to the boundary conditions is $y(t) = t/3$. It is clearly NOT an absolute minimum, not even locally a strong minimum. (We'll leave open for the moment whether it is at least a weak minimum.)

Ok, if you can't bear to be left in the dark: FYI, it's a weak max

12. Natural Boundary Conditions
Assume L as in Problem 10 and assume that $y^* \in C^1([t_0, t_1] \to \mathbb{R}^n)$ is a (weak) minimum for $I[y] := \int_{t_0}^{t_1} L(t, y, \dot{y}) \, dt$, but this time we do NOT prescribe boundary conditions. Show that in this case, on top of the Euler equations, the *natural boundary conditions* $p^*(t_0) = p^*(t_1) = 0$ must hold. (p is the same as in pblm #10)

13. The Catenoid, II:
By means of an independent calculation, find a first order ODE that must necessarily hold for a minimizer r (if one exists), using the lemma from class that was called 'Energy theorem' in anticipation of future discussions. It is a once-integrated version of the Euler-Lagrange equation.

Next solve the Euler-Lagrange equation, showing that the general solution is

$$r(x) = E \cosh \frac{x - x_0}{E} \tag{1}$$

Note: $\cosh t = (e^t + e^{-t})/2$ *and* $\sinh t = (e^t - e^{-t})/2$, $\tanh t = \sinh t / \cosh t$, $\coth t = \cosh t / \sinh t$, *just in case they ruthlessly skipped these perfectly useful functions in calculus.*

Because of the boundary conditions $r(\pm x_1) = r_1$, we have $x_0 = 0$.

Show that there exists exactly one positive number ζ satisfying the equation $\zeta = \coth \zeta$, find its numerical value, and show that there exists no / exactly one / exactly two solutions (1) satisfying the boundary conditions $r(\pm x_1) = r_1$, provided r_1/x_1 is smaller than / equal to / larger than $\sinh \zeta$ (respectively).

14. Absolute Minimizers say 'Farewell' rather than 'See you later'
Assume $\bar{y} : [a, b] \to \mathbb{R}$ und $\tilde{y} : [a, b] \to \mathbb{R}$ are absolute minimizers of a variational problem $I[y] := \int_a^b L(t, y, \dot{y}) \, dt$ (with respect to different boundary conditions). We assume $L_{\dot{y}\dot{y}} > 0$. Show that there cannot exist two points $t_0, t_1 \in [a, b[$ in which $\bar{y}$ und $\tilde{y}$ intersect. Assume as many derivatives of L as you need. *Hint: Otherwise, a minimal with corners could be constructed.*

15. **The Catenoid, III**

We have seen that for r_1/x_1 sufficiently small, there do NOT exist C^1 curves that minimize the area of the rotational surface, because the boundary value problem for the Euler Lagrange equation has no solution. So let us now study the case $r_1/x_1 > \rho := \sinh\zeta$.

Which of the two solutions of the boundary value problem should we proclaim as the best candidate for a minimizer? This is the focus of the present problem.

First show (letting $u := x_1/E$, $s := r_1/x_1$) that the area of the catenoid satisfies:

$$A := \frac{I[r(\cdot,E)]}{2\pi r_1^2} = \frac{u}{\cosh^2 u} + \tanh u$$

with $s = \cosh u/u$. Moreover show that these relations define two decreasing functions $s \mapsto A_\pm(s)$ where

$$\begin{aligned}
&A_+ : [\rho,\infty[\ \to\ ]1,\zeta]\ , \quad A_+(\rho)=\zeta\ ,\ A_+(s)\to 1 \text{ as } s\to\infty\\
&A_- : [\rho,\infty[\ \to\ ]0,\zeta]\ , \quad A_-(\rho)=\zeta\ ,\ A_-(s)\sim 2/s \text{ as } s\to\infty\\
&A_+'(s) > A_-'(s) \text{ for } s>\rho \text{ and therefore } A_+(s) > A_-(s)
\end{aligned}$$

and where A_+ corresponds to the smaller choice for E. (Consider the graphs of $A_\pm$ als parametrized by u.) We conclude therefore that the smaller of the two possible choices of E is disqualified for an absolute minimum. (Later it will transpire that it is even disqualified for a relative minimum.)

There is another interesting curve that leads to a sensible surface of rotation, but was omitted from the original domain: Namely we consider the polygonal path connecting $(-x_1,r_1)\ldots(-x_1,0)\ldots(x_1,0)\ldots(x_1,r_1)$. It is also known under name of Goldschmidt solution. Let us denote its area functional, with the same normalization that was used for $A_\pm$, as $A_\sqcup$. Plot the graph of all three functions A_+, A_-, $A_\sqcup$, and for a representative choice of parameters s, plot also the corresponding curves in the (x,r) plane.

16. **The Catenoid, IV**

Let us begin with an extended, motivating overview of what we know already: For $s = r_1/x_1$ small, namely $s < \rho$, no graph of a C^1 function can qualify as a minimizer, even in the weakest possible sense of a weak minimizer on short segments only. However, in a larger class of curves, we have a natural candidate for a minimizer, namely the Golschmidt solution $r_\sqcup$. It fills separately each of the circles that span the surface, but does not connect these two circles, because they are too far away from each other. However, as the spanning circles move closer, $s > \rho$, one obtains two bona fide curves. The one that hangs lower, r_+, is certainly not an absolute minimizer, because it yields a larger area than its competitor r_-, which hangs higher. r_- MIGHT be an absolute minimizer.

We therefore venture the conjecture that $r_\sqcup$ and r_- are relative (strong) minimizers, whereas r_+ is a saddle point (only short segments are minimals). This is a familiar scenario even in minimax problems in $\mathbb{R}^2$: When tuning through a parameter s, new relative minima may arise at a certain threshold value of s, and other critical points that are not minima arise at the same time. Here is a simple example for this phenomenon in $\mathbb{R}^2$: Take the function I given by $I(x,y) := x + \frac{1}{3}x^3 - sx^2 + y^2$ on $[0,\infty[\times\mathbb{R}$. For $s < 1$, the only minimum is on the boundary, namely $x_\sqcup = 0, y = 0$. But for $s > 1$, another relative minimum $x_- = s + \sqrt{s^2-1}, y = 0$ arises, and there is a saddle point

$x_+ = s - \sqrt{s^2-1}, y = 0$ between the two relative minima. Both come into existence at $s = 1$, and this is where the second derivative $\partial_x^2 I(x,y) = 0$. A little sketch indicates that the latter property is essential for the phenomenon.

Building on this analogy, we hope to find a direction ϕ in function space (i.e., a variation) for the catenoid problem at $s = \rho$, such that the second derivative at $r_+ = r_-$ vanishes in direction ϕ. By a small modification, we hope to find a similar direction ϕ for the case $s > \rho$ such that the second derivative at r_+ in this direction becomes negative. Or at least we want to do this for those $s > \rho$ that are still close to ρ. Having found such a direction, we could know for sure that r_+ is NOT *even a weak minimum. (Conversely, we'll need to await further development of the theory to show that r_- is indeed a strong minimum.)*

Take a fixed $\phi \in C_0^1[-x_*, x_*]$, which we consider as a piecewise C^1 function on the larger interval $[-x_1, x_1] \supset [-x_*, x_*]$ (extended by 0). Show that the second derivative of I in this direction can be written as

$$\left(\frac{d^2}{d\varepsilon^2}\right)\bigg|_{\varepsilon=0} I[r(\cdot,E)+\varepsilon\phi] = 2\pi \int_{-x_*}^{x_*} \left(E\phi'(x)^2 - \frac{1}{E}\phi(x)^2\right)\cosh^{-2}\frac{x}{E}\,dx\ . \qquad (2)$$

We now attempt, using expressions that occur in (2) already, to guess a simple formula for some ϕ that does not vanish until $x = \zeta E$. (Here ζ is the solution to $\zeta = \coth\zeta$.) Such a function ϕ becomes a legitimate variation exactly at $s = \rho$, but fails to fit between the boundary conditions while $s < \rho$. This supports the hope that such a ϕ may make (2) vanish. If we are successful, we try to modifiy the formula slightly, such as to find a legitimate variation for $s > \rho$, and we check whether we can make (2) negative with such a ϕ. A wise but plausible choice of ϕ will succeed on both counts.

Study the roadmap given here, proceeding with the details until eqn. (2), in preparation for a more comprehensive presentation in the lecture.

17. Spherical Pendulum

A mass point is attached to a (weightless) rod so that it can move freely on a sphere, but not leave the sphere. It moves under the force of gravity alone (no friction). In terms of spherical polar coordinates ϑ and φ, derive the equations of motion and the law of conservation of energy, using the Lagrange function method. Here ϑ denotes the angular distance from the zenith position (it is $\pi/2 -$ geographical latitude), and φ denotes the geographical longitude.

Think how you would need to decompose forces into tangential and normal components and transform the accelerations from cartesian coordinates into spherical coordinates, as compared to the ease of writing the SCALAR *quantities kinetic and potential energy in spherical polar coordinates.*

18. The Kepler Problem

A planet of mass m moves in the gravitational potential of a heavy point mass. The variational problem

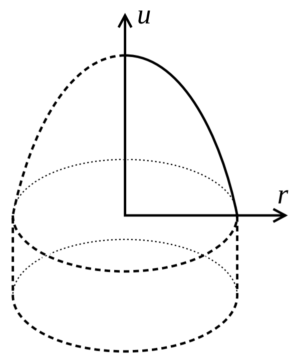
$$I[r,\phi] := \int_{t_0}^{t_1} \left\{\frac{m}{2}(\dot r^2 + r^2\dot\phi^2) + \frac{mG}{r}\right\} dt$$

describes this system. Note how the kinetic and potential energy have been transformed into polar coordinates. Derive the equations of motion and the law of conservation of energy.

Remember that energy is conserved because the Lagrange function $L(t,\phi,r,\dot\phi,\dot r) = \frac{m}{2}(\dot r^2 + r^2\dot\phi^2) + \frac{mG}{r}$ does not actually depend on t. Now have a look at your calculations of the EL equations and notice that L also does not depend on ϕ. This implies that a certain other quantity is a constant. Which quantity? Give it in terms of $(r,\dot\phi,\dot r)$. If you have the physics background, identify the quantity by its name as well.

19. Newton's Air Resistance Model: A Misbehaving Example, from the Point of View of the General Theory

We study the problem to find the surface of a bullet with given cross section (we'll assume a disc of radius R as the cross section) such as to minimize air resistance. Newton suggested the following model:



$$(1)\qquad I[u] := \int_0^R \frac{r\,dr}{1+u'(r)^2}\ .$$

In this model the surface is described as the graph of a radially symmetric piecewise C^1 function $r \mapsto u(r)$.

The physical hypotheses underlying this model are that the air resistance is caused by the exchange of momentum with individual gas molecules 'above' the cross section, as they are hit, and that each gas molecule will be reflected out of the path of the bullet by this impact so that it hits the bullet only once.

We are insisting here that the surface should be rotationally symmetric. It was shown recently that the analogous model without this symmetry requirement (but the same amendments as discussed below for the rotational symmetric case) yields minimizers that are NOT automatically rotationally symmetric.

Firstly show, that the problem needs further assumptions to allow for a solution at all: Very long bullets give $\inf I = 0$. However, even if we require $u \le M$ for some prescribed M, the infimum is still 0. (The idea that worked for long bullets can be salvaged with modification). You may however observe, once you have examples that give arbitrarily small I in spite of the bound $u \le M$, that these examples will NOT abide by the physical modeling assumption underlying the model.

A reasonable and popular hypothesis that should be compatible with the 'one hit per molecule' modeling assumption is as follows: We require $u'(r) \le 0$ und $u \le M$, $u(0) = M$, $u(R) = 0$. It can be shown that under these constraints a minimum does exist and that a minimizing function u is constant M on some interval $[0,a]$ and then has a corner at a. (You are NOT asked to show the claims of this paragraph.)

Obtain Euler's equation and show that a solution to the minimization problem as described in the previous paragraph VIOLATES the Erdmann corner condition. Check the reasoning and hypotheses used for Euler's equation and the corner condition carefully and explain which of the assumptions entering there fails in the present problem. (One assumption must fail to avoid a contradiction.)

Now test the Legendre condition: Show that it is also violated on the interval $[0,a]$. (Failure of the same hypothesis makes this possible.) Show however that the Legendre condition does imply some nontrivial information about the slope (namely?) on the interval $]a,R[$ where u is not constant.

20. **Eigenvalues of symmetric matrices:** Let A be a symmetric $n \times n$ matrix with real entries. We are going to use the Lagrange multiplier technique to prove the following fundamental theomrem from linear algebra: There exists an orthonormal set of n real eigenvectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ of A with corresponding real eigenvalues $\lambda_1, \ldots \lambda_n$, i.e., in formulas:

$$A\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i \quad i = 1, \ldots n$$
$$\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0 \text{ if } i \neq j \ . \quad \boldsymbol{u}_i \cdot \boldsymbol{u}_i = 1$$

We specify the numbering by assuming $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

(a) Show the existence of $(\lambda_1, \boldsymbol{u}_1)$ by means of a direct existence proof for the problem $\min\{\boldsymbol{u}^T A\boldsymbol{u} \mid \boldsymbol{u}^T\boldsymbol{u} = 1\}$, and a Lagrange multiplier argument to find an equation satisfied by the minimizer.

(b) The same argument can be used for $(\lambda_n, \boldsymbol{u}_n)$, with a maximization, but we forego this option, because we want all intermediate eigenvalues as well: Given some $\boldsymbol{u}_1$ from part (a), show the existence of $(\lambda_2, \boldsymbol{u}_2)$ by means of a direct existence proof for the problem $\min\{\boldsymbol{u}^T A\boldsymbol{u} \mid \boldsymbol{u}^T\boldsymbol{u} = 1\ ,\ \boldsymbol{u} \cdot \boldsymbol{u}_1 = 0\}$, and another Lagrange multiplier argument.

(c) The method can be repeated to find $(\lambda_3, \boldsymbol{u}_3)$ from the minimum problem $\min\{\boldsymbol{u}^T A\boldsymbol{u} \mid \boldsymbol{u}^T\boldsymbol{u} = 1\ ,\ \boldsymbol{u} \cdot \boldsymbol{u}_1 = 0\ ,\ \boldsymbol{u} \cdot \boldsymbol{u}_2 = 0\}$, etc., until all eigenvalues are constructed.

This is the most fundamental application of the multivariable calculus version of Lagrange multipliers. It lends itself to a natural generalization in calculus of variations, to prove the existence of solutions for eigenvalue problems in ODEs (in particular Fourier analysis as a special case) and PDEs, and it can even be used to extract some properties of the eigenfunctions.

21. **Minimax and Maximin:** In the previous problem, it is sometimes inconvenient that one has to construct the eigenvalues recursively, in particular that information about $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{i-1}$ enters into the construction of $(\lambda_i, \boldsymbol{u}_i)$. This deficit can be mended by showing the following:

$$\lambda_i = \min\left\{ \max\left\{ \boldsymbol{u}^T A\boldsymbol{u} \;\middle|\; \boldsymbol{u}^T\boldsymbol{u} = 1\ ;\ \boldsymbol{u} \in E \right\} \;\middle|\; E \text{ an } i \text{ dim. subspace of } \mathbb{R}^n \right\}$$

$$\lambda_i = \max\left\{ \min\left\{ \boldsymbol{u}^T A\boldsymbol{u} \;\middle|\; \boldsymbol{u}^T\boldsymbol{u} = 1\ ;\ \boldsymbol{u} \perp E \right\} \;\middle|\; E \text{ an } i-1 \text{ dim. subspace of } \mathbb{R}^n \right\}$$

22. **Sturm Comparison Principle; Catenoid V:** The Sturm comparison principle is designed for scalar, (primarily) linear, ODEs of 2nd order, occasionally also for differential inequalities. Assume φ and ψ solve the same ODE (or, occasionally, very similar ODEs). One considers the Wronskian $W := \varphi'\psi - \varphi\psi'$, uses the ODE(s) to obtain information about W' and uses this information in $\int W'$ to draw conclusions about the relative location of zeros of φ and ψ.

Use this principle to show the following: If φ solves the ODE $u''(y) - \tanh y u'(y) + u(y) = 0$ subject to the boundary conditions $\varphi(y_-) = \varphi(y_+) = 0$ and $\varphi > 0$ in $]y_-, y_+[$, and if ψ solves the same ODE (no boundary information) with $\psi(y_0) = 0$ for some $y_0 \in]y_-, y_+[$, then ψ cannot have another zero in the closed interval $[y_-, y_+]$, unless $\psi \equiv 0$. Use this result to show, by Jacobi's criterion, that the 'flat' catenoid r_- is (at least) a weak minimum. (You may need to absorb or adjust a few constants to match the two parts of the problem.)

23. **Catenoid VI:** Carry out the calculation outlined in the notes to show, by means of extremal fields and the Weierstrass $\mathfrak{E}$ function, that the segment r_- is indeed a strong minimum with respect to symmetric boundary conditions $r(\pm x_1) = r_1$.

Find some explicit threshold θ such that for $r_1/x_1 > \theta$, r_- can be shown to be an absolute minimum. This could be done according to the following principle: A certain ('sufficiently large') neighbourhood of the graph of r_- can be covered with nonintersecting extremals, such as to guarantee that r_- yields the smallest area among those curves that stay inside the neighbourhood; an explicit estimate shows that any curve leaving this neighbourhood must of necessity give a larger area than r_-. Competition: who can find the smallest valid θ by this method? *Of course, if an existence proof can be obtained by a nontrivial variant of direct methods, one can get a sharp result, because then only the three candidates discussed previously need to be compared.*

24. **Family of Extremals:** Assuming sufficient differentiability we assume that $y(t;b)$ is a family of extremals, i.e., for each b, the function $y(\cdot,b)$ solves the EL equation. We make no hypotheses on boundary conditions. Distinguish $\dot{y} = \partial y/\partial t$ from $y' = \partial y/\partial b$.

With the obvious notation

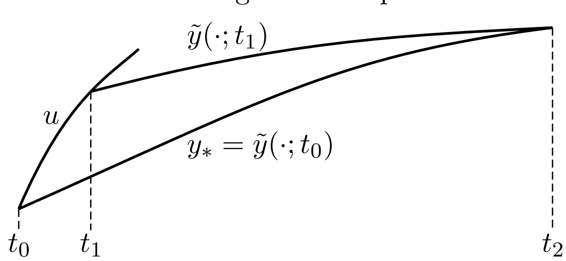
$$I_{t_0}^{t_1}[y(\cdot,b)] := \int_{t_0}^{t_1} L(t, y(t,b), \dot{y}(t,b))\, dt\ .$$

show that

$$\frac{\partial}{\partial b} I_{t_0}^{t_1}[y(\cdot,b)] = \big[L_{\dot{y}}(t, y(t,b), \dot{y}(t,b)) y'(t,b)\big]_{t_0}^{t_1}$$

25. **Strong Minimals and the Weierstrass $\mathfrak{E}$-Function:**
Here we derive a *necessary* condition for a strong minimum. To this end, 'weak' variations of the type $y_* \to y_* + \varepsilon\varphi$ do not suffice, because for small ε the comparison functions would automatically be in a *narrow* neighbourhood. So instead we need to invent comparison curves which, for $\varepsilon \to 0$, converge to y_* in such a way that the derivatives do not converge; at least they must fail to converge in some point.

We do this by considering the following variation $\hat{y}$ of the segment y_* on the interval $[t_0, t_2]$: For any curve segment u with $u(t_0) = y_*(t_0)$ we let $\hat{y} = u$ on $[t_0, t_1]$, and $\hat{y} = \tilde{y}(\cdot; t_1)$ on $[t_1, t_2]$, where $\tilde{y}$ $\tilde{y}(t_1; t_1) = u(t_1)$ and $\tilde{y}(t_2; t_1) = y_*(t_2)$.



For the moment, we don't worry about the question how to construct such a segment $\tilde{y}$. (It can be shown that a unique such segment exists for t_1 sufficiently close to t_0, provided conjugate points don't exist. And they don't if the strict Legendre condition is satisfied and t_2 is sufficiently close to t_0.)

If y_* is a strong minimal over the interval $[t_0, t_2]$, then obviously it must hold:

$$\frac{d}{dt_1}\Big(I_{t_0}^{t_1}[u] + I_{t_1}^{t_2}[\tilde{y}(\cdot;t_1)]\Big) \geq 0 \ \text{ at } t_1 = t_0 \ .$$

(a) Explain why we only get an inequality rather than an equality.

(b) Carrying out the derivative and using the previous problem, conclude that for strong minimality of short segments (near t_0), it is necessary that the Weierstrass condition

$$\begin{aligned} &\mathfrak{E}(t_0, y_*(t_0), \dot{y}_*(t_0), \dot{Y}) \geq 0 \quad \forall\, \dot{Y} \in \mathbb{R}^n \qquad \text{with} \\ &\mathfrak{E}(t, y, \dot{y}, \dot{Y}) := L(t, y, \dot{Y}) - L(t, y, \dot{y}) - L_{\dot{y}}(t, y, \dot{y})(\dot{Y} - \dot{y}) \end{aligned}$$

be verified.

26. Brachystochrone again:
Use the method of extremal fields to show that the brachystochrone is indeed an absolute minimal.