

**Examples Jordan Normal Form**  
**UTK – M531 – Ordinary Differential Equations I**  
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(These examples were generated with the help of symbolic algebra software, and also the calculations were done using such software.)

$\text{diag}(\lambda_1, \dots, \lambda_n)$  denotes the diagonal matrix (size  $n \times n$ ) with the diagonal entries as specified.

I may write column vectors like  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  as transposed rows  $[1, 2, 3]^T$  for typographical reasons.

**Example 1:**

The matrix

$$A = \begin{bmatrix} \frac{44}{3} & -\frac{82}{3} & -20 \\ \frac{19}{3} & -\frac{35}{3} & -10 \\ 4 & -8 & -4 \end{bmatrix}$$

has characteristic polynomial

$$\det(A - \lambda I) = -\lambda^3 - \lambda^2 + 10\lambda - 8 = -(\lambda - 2)(\lambda - 1)(\lambda + 4)$$

where the factorization is based on eyeballing (guesswork) to find  $\lambda = 1$  as one root, and then the quadratic formula.

Since all eigenvalues are distinct, we can diagonalize the matrix  $A = SDS^{-1}$  with  $D = \text{diag}(1, 2, -4)$ . (Any other order like eg.  $\tilde{D} = \text{diag}(1, -4, 2)$  would do just as well, with a different  $\tilde{S} \neq S$  (similarly permuted). So we have chosen the numbering  $\lambda_1 = 1$ ,  $\lambda_2 = 2$ ,  $\lambda_3 = -4$ . Let's find corresponding eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ :

$\mathbf{v}_1$  is calculated as a solution to the linear system  $(A - \lambda_1)\mathbf{v}_1 = \mathbf{0}$ :

$$\begin{aligned} \frac{41}{3}v_1^{(1)} + \frac{82}{3}v_1^{(2)} + (-20)v_1^{(3)} &= 0 \\ \frac{19}{3}v_1^{(1)} + \frac{-38}{3}v_1^{(2)} + (-10)v_1^{(3)} &= 0 \\ 4v_1^{(1)} + (-8)v_1^{(2)} + (-5)v_1^{(3)} &= 0 \end{aligned}$$

(Upper indices denote components of the vector, lower indices identify vectors.) Gauss elimination from this system produces:  $v_1^{(1)} = 2v_1^{(2)}$ ,  $v_1^{(3)} = 0$ . We may therefore choose  $\mathbf{v}_1 = [2, 1, 0]^T$ . Any nonzero multiple would have been just as legitimate a choice (and would have led to a somewhat different matrix  $S$ ).

Likewise we obtain  $\mathbf{v}_2 = [1, -1, 2]^T$  and  $\mathbf{v}_3 = [4, 2, 1]^T$ , where again nonzero multiples would have been just as legitimate choices. We have therefore found (our choice of) the matrix  $S$ :

$$S = \left[ \begin{array}{cc|c} 2 & 1 & 4 \\ 1 & -1 & 2 \\ 0 & 2 & 1 \end{array} \right]$$

(Vertical lines inserted to make eigenvectors more clearly visible). It is another bunch of

linear equations to find its inverse (if you need it):

$$S^{-1} = \frac{1}{3} \begin{bmatrix} 5 & 7 & -6 \\ 1 & -2 & 0 \\ -2 & 4 & 3 \end{bmatrix}$$

You can now check explicitly that  $AS = SD$ , or  $A = SDS^{-1}$ .

**Example 2:**

This example must be higher dimensional, because its purpose is to display several bells and whistles at the same time. Took me quite a while to come up with an example with *somewhat* decent numbers. Let

$$A = \begin{bmatrix} 28 & 4 & -1 & 1 & -9 & -15 & -8 \\ 90 & 4 & -2 & 2 & -36 & -45 & -27 \\ 93 & 0 & -2 & 0 & -40 & -45 & -28 \\ 93 & 0 & 0 & -2 & -40 & -45 & -28 \\ -560 & -12 & 3 & -3 & 235 & 275 & 169 \\ -554 & 4 & -1 & 1 & 243 & 267 & 168 \\ 1790 & 24 & -6 & 6 & -759 & -875 & -540 \end{bmatrix}$$

The characteristic polynomial is

$$\det(A - \lambda I) = -\lambda^7 - 10\lambda^6 - 41\lambda^5 - 90\lambda^4 - 120\lambda^3 - 112\lambda^2 - 80\lambda - 32 = -(\lambda + 2)^5(\lambda^2 + 1)$$

where the factorization is a conspiratively designed lucky ‘coincidence’. So  $\lambda_1 = -2$  is an eigenvalue with algebraic multiplicity 5 and  $\lambda_{6,7} = \pm i$  are single eigenvalues. To find the geometric multiplicity of  $\lambda_1$ , we must actually solve the linear system  $(A + 2I)\mathbf{v} = \mathbf{0}$ . Let me do this ‘by hand’, with ‘unsystematic’ row transformations (i.e., smart pivoting) to preserve nice numbers:

30	4	-1	1	-9	-15	-8		30	4	-1	1	-9	-15	-8
90	6	-2	2	-36	-45	-27		-3	6	-2	2	4	0	1
93	0	0	0	-40	-45	-28		93	0	0	0	-40	-45	-28
93	0	0	0	-40	-45	-28	step1	0	0	0	0	0	0	0
-560	-12	3	-3	237	275	169	→	-560	-12	3	-3	237	275	169
-554	4	-1	1	243	269	168		6	16	-4	4	6	-6	-1
1790	24	-6	6	-759	-875	-538		110	-12	3	-3	-48	-50	-31

step 1: subtract 3rd row from 2nd and 4th; subtract 5th row from 6th and add three times to 7th.

step 2: add 5×last row to 5th, then subtract 3rd row from last, then use 2nd row to produce leading 0’s in 1st, 3rd, 6th row

step 3: move 4th row to bottom, 2nd row to top, and use it to create more zeros in the 1st column

0	64	-21	21	31	-15	2		-3	6	-2	2	4	0	1
-3	6	-2	2	4	0	1		0	64	-21	21	31	-15	2
0	186	-62	62	84	-45	3		0	186	-62	62	84	-45	3
0	0	0	0	0	0	0	step3	0	-92	$\frac{74}{3}$	$-\frac{74}{3}$	$-\frac{49}{3}$	25	$\frac{32}{3}$
-10	-72	18	-18	-3	25	14	→	0	28	-8	8	14	-6	1
0	28	-8	8	14	-6	1		0	22	$-\frac{25}{3}$	$\frac{25}{3}$	$\frac{44}{3}$	-5	$\frac{8}{3}$
17	-12	3	-3	-8	-5	-3		0	0	0	0	0	0	0

step 4: subtract 2nd row 3 times from 3rd, add it to 4th, subtract 5th row from 6th

step 5: add 5th row to 4th, subtract it 5/2 times from 2nd; subtract 3rd row from 6th

$$\begin{array}{cccccc|cccccc}
 -3 & 6 & -2 & 2 & 4 & 0 & 1 & -3 & 6 & -2 & 2 & 4 & 0 & 1 \\
 0 & 64 & -21 & 21 & 31 & -15 & 2 & 0 & -6 & -1 & 1 & -4 & 0 & -\frac{1}{2} \\
 0 & -6 & 1 & -1 & -9 & 0 & -3 & 0 & -6 & 1 & -1 & -9 & 0 & -3 \\
 0 & -28 & \frac{11}{3} & -\frac{11}{3} & \frac{44}{3} & 10 & \frac{38}{3} & 0 & 0 & -\frac{13}{3} & \frac{13}{3} & \frac{86}{3} & 4 & \frac{41}{3} \\
 0 & 28 & -8 & 8 & 14 & -6 & 1 & 0 & 28 & -8 & 8 & 14 & -6 & 1 \\
 0 & -6 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 1 & \frac{5}{3} & 0 & 0 & -\frac{4}{3} & \frac{4}{3} & \frac{29}{3} & 1 & \frac{14}{3}
 \end{array} \xrightarrow{\text{step5}}$$

step 6: multiply 4th and 6th row by 3; use 3rd row to produce 0's in column2 of 2nd and 5th row

step 7: exchange 3rd and 2nd row; use new 3rd (old 2nd) row to produce 0's in 3rd column

$$\begin{array}{cccccc|cccccc}
 -3 & 6 & -2 & 2 & 4 & 0 & 1 & -3 & 6 & -2 & 2 & 4 & 0 & 1 \\
 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} & 0 & -6 & 1 & -1 & -9 & 0 & -3 \\
 0 & -6 & 1 & -1 & -9 & 0 & -3 & 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} \\
 0 & 0 & -13 & 13 & 86 & 12 & 41 & 0 & 0 & 0 & 0 & \frac{107}{2} & 12 & \frac{99}{4} \\
 0 & 0 & -\frac{10}{3} & \frac{10}{3} & -28 & -6 & -13 & 0 & 0 & 0 & 0 & -\frac{109}{3} & -6 & -\frac{103}{6} \\
 0 & 0 & -4 & 4 & 29 & 3 & 14 & 0 & 0 & 0 & 0 & 19 & 3 & 9
 \end{array} \xrightarrow{\text{step7}}$$

step 8: multiply row5 by 3/2 and add row 4 to it: it becomes (0,0,0,0,-1,3,-1); use it to produce 0's in column 5.

step 9: the 4th row has become a multiple of the 6th and can be dropped; normalize the 6th row.

$$\begin{array}{cccccc|cccccc}
 -3 & 6 & -2 & 2 & 4 & 0 & 1 & -3 & 6 & -2 & 2 & 4 & 0 & 1 \\
 0 & -6 & 1 & -1 & -9 & 0 & -3 & 0 & -6 & 1 & -1 & -9 & 0 & -3 \\
 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} & 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} \\
 0 & 0 & 0 & 0 & 0 & \frac{345}{2} & -\frac{113}{4} & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\
 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 6 & -1 \\
 0 & 0 & 0 & 0 & 0 & 60 & -10 & 0 & 0 & 0 & 0 & 0 & 6 & -1
 \end{array} \xrightarrow{\text{step9}}$$

We therefore find two eigenvectors:  $v^{(7)}$  can be chosen arbitrarily, but the 5th eqn requires  $6v^{(6)} - v^{(7)} = 0$ . The 4th eqn then determines  $v^{(5)}$ , whereas  $v^{(4)}$  is again arbitrary. The other components are then determined again. The eigenspace is therefore 2-dimensional (geometric multiplicity 2), and we choose two linearly independent eigenvectors  $\mathbf{v}$  and  $\mathbf{w}$  by letting  $v^{(7)} = 2, v^{(4)} = 0$ , and  $w^{(7)} = 0, w^{(4)} = 1$ . Other choices would be equally legitimate and would simply produce a different matrix  $S$ .

We thus find the eigenvectors  $\mathbf{v} = [\frac{1}{3}, \frac{1}{2}, 0, 0, -1, \frac{1}{3}, 2]^T$  and  $\mathbf{w} = [0, 0, 1, 1, 0, 0, 0]^T$ .

We try to find further vectors by solving  $(A + 2I)\mathbf{v}' = \mathbf{v}$  and  $(A + 2I)\mathbf{w}' = \mathbf{w}$  (if possible). And then, we try to solve  $(A + 2I)\mathbf{v}'' = \mathbf{v}'$  and  $(A + 2I)\mathbf{w}'' = \mathbf{w}'$  (if possible). We wouldn't know yet if any particular among these equations has a solution, but the JNF theorem guarantees that altogether we find three solutions, i.e., among the tentative vectors  $\{\mathbf{v}', \mathbf{v}'', \mathbf{v}''', \dots, \mathbf{w}', \mathbf{w}'', \mathbf{w}''', \dots\}$  three will actually exist (such as to bring the total to 5 generalized eigenvectors, according to the algebraic multiplicity 5).

I skip the remaining linear systems (the row transformations could be reused) and merely give the results:

The system  $(A + 2I)\mathbf{v}' = \mathbf{v}$  has the solutions  $\mathbf{v}' = [0, \frac{1}{12}, 0, 0, 0, 0, 0]^T + c_1\mathbf{v} + c_2\mathbf{w}$ . The most convenient choice is of course  $c_1 = c_2 = 0$ , and we have  $\mathbf{v}' = [0, \frac{1}{12}, 0, 0, 0, 0, 0]^T$ .

The system  $(A + 2I)\mathbf{w}' = \mathbf{w}$  has the solutions  $\mathbf{w}' = [2, \frac{3}{2}, 0, 0, -1, 5, 0]^T + c_3\mathbf{v} + c_4\mathbf{w}$ , and again we choose  $\mathbf{w}' = [2, \frac{3}{2}, 0, 0, -1, 5, 0]^T$ .

The system  $(A + 2I)\mathbf{v}'' = \mathbf{v}'$  has the solutions  $\mathbf{v}'' = [\frac{1}{4}, \frac{1}{3}, \frac{-1}{6}, 0, \frac{-3}{4}, \frac{1}{4}, \frac{3}{2}]^T + c_5\mathbf{v} + c_6\mathbf{w}$ . We choose  $c_5 = c_6 = 0$ .

The system  $(A + 2I)\mathbf{w}'' = \mathbf{w}'$  has NO solutions.

We have already 5 vectors, and indeed the system  $(A + 2I)\mathbf{v}''' = \mathbf{v}''$  (which we might yet consider trying) has no solutions. At this moment we know that the two Jordan blocks for  $\lambda = -2$  have sizes 3 and 2 respectively, corresponding to the sets of vectors  $\{\mathbf{v}_1 := \mathbf{v}, \mathbf{v}_2 := \mathbf{v}', \mathbf{v}_3 := \mathbf{v}''\}$  and  $\{\mathbf{v}_4 := \mathbf{w}, \mathbf{v}_5 := \mathbf{w}'\}$ . (Now that we know the sizes of the Jordan blocks, we can number them sensibly.)

We still need eigenvectors  $\mathbf{v}_6$  and  $\mathbf{v}_7$  for the eigenvalues  $i$  and  $-i$  respectively. Then we put them all as columns in our matrix  $S$ . We get  $AS = SJ$  with :

$$S = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{4} & 0 & 2 & -1 & -1 \\ \frac{1}{2} & \frac{1}{12} & \frac{1}{3} & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{6} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & \frac{-3}{4} & 0 & -1 & -1+i & -1-i \\ \frac{1}{3} & 0 & \frac{1}{4} & 0 & 5 & \frac{-9+8i}{5} & \frac{-9-8i}{5} \\ 2 & 0 & \frac{3}{2} & 0 & 0 & 1-4i & 1+4i \end{bmatrix} \quad J = \begin{bmatrix} -2 & 1 & 0 & | & & & \\ 0 & -2 & 1 & | & & & \\ 0 & 0 & -2 & | & 0 & & \\ \hline & & & & -2 & 1 & \\ & & & & 0 & -2 & \\ \hline & & & & & & i \\ \hline & & & & & & -i \end{bmatrix}$$

$S$  is invertible as predicted by the general theory. I won't write down  $S^{-1}$ , but note that  $\det S = i/540$ , therefore  $A = SJS^{-1}$ .

$\ker(A + 2I)$  is spanned by columns 1 and 4 of  $S$ ,  $\ker(A + 2I)^2$  is spanned by columns 1,2,4,5 of  $S$ . The complete eigenspace for eigenvalue  $-2$  is  $\ker(A + 2I)^3 = \ker(A + 2I)^j$  for any  $j \geq 3$ , spanned by columns 1-5 of  $S$ . — Have a look at the (3,4) entry of  $J$ , which is 0. It is this 0 that distinguishes between the case of a 3-Jordan block and a 2-Jordan block, as opposed to a single 5-Jordan block.

The span of columns 6 and 7 of  $S$  can also be spanned by real vectors (which are then no longer eigenvectors), namely the real and imaginary parts of these two vectors,  $[-1, 0, 0, 0, -1, -\frac{9}{5}, 1]^T$  and  $[0, 0, 0, 0, 1, \frac{8}{5}, -4]^T$ .

## Exponentials

The exponential of a Jordan block of size  $n$  can be calculated explicitly:

$$\exp t \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & \\ & & \ddots & 1 & \\ & & & & \lambda \end{bmatrix} = \exp \left( t\lambda I + t \begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & & 0 \end{bmatrix} \right) = e^{t\lambda} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & t \end{bmatrix}$$

Therefore the JNF theorem implies that  $\exp(tA) \rightarrow 0$  as  $t \rightarrow \infty$  *if and only if* all eigenvalues of  $A$  have  $\operatorname{Re} \lambda < 0$ .

Moreover,  $\exp(tA)$  remains bounded as  $t \rightarrow \infty$  *if and only if* all eigenvalues of  $A$  have  $\operatorname{Re} \lambda \leq 0$  *and* all Jordan blocks for eigenvalues with  $\operatorname{Re} \lambda = 0$  have size 1.