Homework Sample Solutions
UTK – M447 – Honors Advanced Calculus I – Fall 2015
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1: Prove that for every $a \in \mathbb{R}_+$, there exists a unique $x \in \mathbb{R}_+$ for which $x^2 = a$. Specifically:
(a) Show uniqueness of x; also obtain the lemma "If $x, y > 0$ and $x^2 = a$ and $y^2 = b$, and $a < b$, then $x < y$" in this proof.
(b) Letting $S := \{y \in \mathbb{R} \mid y > 0 \text{ and } y^2 \leq a\}$, show that S is non-empty and bounded above.
(c) Defining $x := \sup S$, show that for every $\varepsilon > 0$, $a - \varepsilon < x^2 < a + \varepsilon$, then conclude $x^2 = a$.

Solution:
(a) We show:

$$(*) \qquad \text{If } 0 < x < y, \text{ then } x^2 < y^2.$$

Indeed, we may multiply the inequality $x < y$ with the positive quantity x and get $x^2 < xy$. We also may multiply $x < y$ with the positive quantity y and get $xy < y^2$. By transitivity, we obtain $x^2 < y^2$.

From $(*)$, uniqueness follows: Assuming $x^2 = y^2 = a$ and $x, y > 0$, we can rule out $x < y$ (which would imply $x^2 < y^2$) and $y < x$ (which would imply $y^2 < x^2$). So by trichotomy, only the possibility $x = y$ remains.

Trichotomy with $(*)$ also gives the converse statement "If $x^2 < y^2$ and $x, y > 0$, then $x < y$." For, if x were $= y$, we would conclude that $x^2 = y^2$; if x were $> y$, we would conclude $x^2 > y^2$. Either conclusion violates the hypothesis $x^2 < y^2$. So by trichotomy, only $x < y$ remains.

Note: Uniqueness alone can also be proved differently. Assuming $x^2 = y^2$, we can write this as $x^2 - y^2 = 0$, or equivalently $(x - y)(x + y) = 0$. Using the property[1] in a field that a product can only be 0 if one factor is 0, we conclude that $x - y = 0$ or $x + y = 0$. But $x > 0$ and $y > 0$ implies $x + y > 0$, so the only remaining possibility is $x = y = 0$, i.e., $x = y$.

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[1] The lemma used here, namely "If $uv = 0$ then $u = 0$ or $v = 0$" follows from the field axioms easily: If $u = 0$, we are done. If $u \neq 0$, there exists a multiplicative inverse u^{-1}, and by multiplying $uv = 0$ with u^{-1}, we find $v = 0 \cdot u^{-1} = 0$, and we are again done.

(b) If $a \geq 1$, then $y = 1$ is in S. If $a < 1$ (but $a > 0$ per assumption $a \in \mathbb{R}_+$), then $a \in S$ because $a^2 < a \cdot 1 = a < 1$. Either way, $S \neq \emptyset$.
To show that S is bounded above, we argue: If $a \leq 1$, then 1 is an upper bound for S, because $y > 1$ would imply $y^2 > 1^2 = 1$, contradicting $y^2 \leq a \leq 1$. If however $a > 1$, then a is an upper bound, because $y > a$ would imply $y^2 > a^2 > a \cdot 1 = a$.

Note: We can summarize by saying $\min\{a, 1\} \in S$, and $\max\{a, 1\}$ is an upper bound for S.

(c) By part (b), S has a supremum. Let $x := \sup S$, and let $\varepsilon > 0$. We know $x > 0$ because x is an upper bound for a set S containing the positive number $\min\{a, 1\}$.

To show $x^2 < a + \varepsilon$ we use that x is the *smallest* upper bound, i.e., that for any $\delta > 0$, the number $x - \delta$ is not an upper bound any more. We'll specify δ shortly, but commit to $\delta < x$ already. There exists an $y \in S$ such that $y > x - \delta > 0$. Then, from $(*)$ in part (a) we have $y^2 > (x - \delta)^2 = x^2 - 2x\delta + \delta^2$. This implies $y^2 > x^2 - 2x\delta$. Since $y \in S$, we have $y^2 \leq a$ also; together we conclude $x^2 - 2x\delta < a$. Now choosing $\delta \leq \frac{\varepsilon}{2x}$, we conclude $x^2 < a + 2x\delta \leq a + \varepsilon$.

To show $x^2 > a - \varepsilon$, we use that x is an upper bound for S and assume, for the sake of contradiction, that $x^2 \leq a - \varepsilon$. We intend to exhibit $\delta > 0$ such that $x + \delta \in S$, contradicting x being an upper bound for S. To this end we calculate $(x+\delta)^2 = x^2 + 2x\delta + \delta^2$. Let's require that $\delta \leq x$ and also $\delta \leq \frac{\varepsilon}{3x}$, namely we can take $\delta := \min\{x, \frac{\varepsilon}{3x}\}$. Then

$$(x+\delta)^2 = x^2 + 2x\delta + \delta^2 \leq a - \varepsilon + 2x\delta + \delta \cdot x = a - \varepsilon + 3x\delta \leq a - \varepsilon + 3x\frac{\varepsilon}{3x} = a ,$$

so $x + \delta \in S$.

We have thus proved: $\forall \varepsilon > 0 : a - \varepsilon < x^2 < a + \varepsilon$. To conclude $x^2 = a$, we rule out the other two possibilities $x^2 > a$ and $x^2 < a$. Indeed, if x^2 were $> a$, then we could choose $\varepsilon := x^2 - a > 0$ and conclude $x^2 < a + \varepsilon = a + (x^2 - a)$, hence $x^2 < x^2$, a contradiction. If in contrast, x^2 were $< a$, then we could choose $\varepsilon := a - x^2 > 0$ and conclude $x^2 > a - \varepsilon = a - (a - x^2)$, hence $x^2 > x^2$, again a contradiction.

2: For $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, define $|z| := \sqrt{x^2 + y^2}$. Prove that $|z + w| \leq |z| + |w|$. Make sure that any two inequalities you write down are properly connected with a logical direction of implication (verbally or in symbols: $\Longrightarrow$ or $\Longleftarrow$ or $\Longleftrightarrow$), and that the logical direction of implication matches the needs for the proof.

You may only use ordered field axioms, simple conclusions from them (like subtraction on both sides of inequalities, or 'a < b and c < d implies a + c < b + d' or '0 < a < b and 0 < c < d implies 0 < ab < cd' and their analogs with ≤), the supremum axiom and simple conclusions thereof, and results from the previous problem about the square root.

Solution:
Note beforehand: From the previous problem, we will use the existence and uniquenness of a nonnegative square root for any nonnegative real number. It was proved for 'positive' instead of nonnegative. Including 0 is an easy consequence.

Letting $z = x + iy$ and $w = u + iv$, where $x, y, u, v \in \mathbb{R}$, we want to prove

$$\sqrt{(x+u)^2 + (y+v)^2} \leq \sqrt{x^2 + y^2} + \sqrt{u^2 + v^2} .$$

Since both sides are nonnegative, it is sufficient (and necessary – but the sufficiency part is relevant) to prove

$$\left(\sqrt{(x+u)^2 + (y+v)^2}\right)^2 \leq \left(\sqrt{x^2 + y^2} + \sqrt{u^2 + v^2}\right)^2 ,$$

or equivalently

$$(x+u)^2 + (y+v)^2 \leq (x^2 + y^2) + 2\sqrt{x^2 + y^2}\sqrt{u^2 + v^2} + (u^2 + v^2) .$$

Subtracting $x^2 + y^2 + u^2 + v^2$, and dividing by 2, this is equivalent to

$$xu + yv \leq \sqrt{x^2 + y^2}\sqrt{u^2 + v^2} .$$

Since the right hand side is nonnegative, it is sufficient (but not necessary – eg the left hand side could be negative) to prove

$$(xu + yv)^2 \leq \left(\sqrt{x^2 + y^2}\sqrt{u^2 + v^2}\right)^2 .$$

This is equivalent to

$$x^2u^2 + 2xuyv + y^2v^2 \le (x^2 + y^2)(u^2 + v^2)\ ,$$

or, by subtracting the left side and refactoring on the right

$$0 \le (xv - yu)^2$$

which is known to be a true statement. (Squares are all nonnegative).

Note: An unmotivated writeup could start at the end and argue with 'therefore' in each step. For the writeup as presented, which has the advantage of being motivated, it is crucial that the words 'sufficient' or 'equivalent' (whichever is applicable), or synonymous wordings thereof, connect the statements to make sure the logical direction is correct.

3: Prove: If $\lim a_n = a_*$ and $\lim b_n = b_*$ and $b_* \neq 0$, then $\lim(a_n/b_n) = a_*/b_*$.

Solution: Assume $\lim a_n = a_*$ and $\lim b_n = b_*$ and $b_* \neq 0$.

We now let $\varepsilon > 0$, and assume, without loss of generality, that $\varepsilon \le \min\{|b_*|/2, 1\}$.

There exists then n_0 such that for all $n \ge n_0$, it holds $|a_n - a_*| < \varepsilon$, and $|b_n - b_*| < \varepsilon$; this implies in particular that $|b_n| \ge |b_*| - |b_n - b_*| > |b_*|/2$ and $|a_n| \le |a_*| + |a_n - a_*| < |a_*| + 1$.

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{a_*}{b_*}\right| &= \left|\frac{a_n}{b_n} - \frac{a_n}{b_*} + \frac{a_n}{b_*} - \frac{a_*}{b_*}\right| \le \left|\frac{a_n}{b_n} - \frac{a_n}{b_*}\right| + \left|\frac{a_n}{b_*} - \frac{a_*}{b_*}\right| \\ &= \left|\frac{a_n(b_* - b_n)}{b_* b_n}\right| + \left|\frac{a_n - a_*}{b_*}\right| \le \frac{|a_*| + 1}{|b_*|^2/2}\varepsilon + \frac{\varepsilon}{|b_*|} =: M\varepsilon\ .\end{aligned}$$

We have thus shown: for every $\varepsilon > 0$ there exists n_0 such that for $n \ge n_0$ it holds $|\frac{a_n}{b_n} - \frac{a_*}{b_*}| < M\varepsilon$ with M determined by a_*, b_* (independent of ε).

Should M be larger than 1, we apply this result, for given $\varepsilon > 0$ to ε/M.

Note: There are of course many variants. One could restrict to proving the special case where $a_n = 1$ for all n, and then appeal to the already proved product rule.

4: Prove (directly from the axioms and consequences proved in class) that $\lim \frac{1}{n} = 0$. (That should be just a few lines.)

Solution: Let $\varepsilon > 0$. We have to find n_0 such that for $n \geq n_0$, it holds $-\varepsilon < \frac{1}{n} < \varepsilon$. Let n_0 be a natural number larger than $\frac{1}{\varepsilon}$. The existence of such an n_0 is guaranteed by the Archimedean property, as a consequence from the supremum axiom, as proved in class. Then, for $n \geq n_0$, we have $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$. On the other side, trivially $\frac{1}{n} > 0 > -\varepsilon$ for all n.

5: Assuming (a_n) and (b_n) are bounded sequences of real numbers: Is the following statement true or false? "¿$\limsup(a_n+b_n) = \limsup a_n + \limsup b_n$?" If true, provide a proof; if false, provide a counterexample, and if possible, prove an amendment replacing '=' with either '≤' or '≥' that results in a true statement. If no such amendment can be proved, provide counterexamples against the amended versions.

Solution: The equality is false. For instance, let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $a_n + b_n = 0$. Now $\limsup(a_n + b_n) = 0$, whereas $\limsup a_n = \limsup b_n = 1$.

We prove the amended version

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n \ ,$$

in other words:

$$\lim_{k\to\infty} \sup\{a_n + b_n : n \geq k\} \leq \lim_{k\to\infty} \sup\{a_n : n \geq k\} + \lim_{k\to\infty} \sup\{b_n : n \geq k\} \ .$$

It suffices to show:

$$\sup\{a_n + b_n : n \geq k\} \leq \sup\{a_n : n \geq k\} + \sup\{b_n : n \geq k\} \ ,$$

because the limit $\lim_{k\to\infty}$ preserves nonstrict inequalities. Since $\hat{a}_k := \sup\{a_n : n \geq k\}$ satisfies $\hat{a}_k \geq a_n$ for all $n \geq k$ and likewise $\hat{b}_k := \sup\{b_n : n \geq k\}$ satisfies $\hat{b}_k \geq b_n$ for all $n \geq k$, we obtain $a_n + b_n \leq \hat{a}_k + \hat{b}_k$ for all $n \geq k$. So $\hat{a}_k + \hat{b}_k$ is an upper bound for the set $\{a_n + b_n : n \geq k\}$, and therefore it is $\geq$ the supremum of that set.

6: Prove by induction the lemma: If $y > -1$ and $n \in \mathbb{N}$, then $(1 + y)^n \geq 1 + ny$. (You'll need it later.)

Solution: The claim is trivially true for $n = 1$. Assuming it true for some arbitrary $n \in \mathbb{N}$, we have to conclude that it is also true for $n + 1$:

$$(1 + y)^{n+1} = (1 + y)^n(1 + y) \geq_{(1)} (1 + ny)(1 + y) = 1 + (n + 1)y + y^2 \geq 1 + (n + 1)y \ ;$$

In step (1), the induction hypothesis and the fact that $1 + y > 0$ was used.

7: (a) Given $x \in \mathbb{R}$, consider the sequence (a_n) given by $a_n := (1 + \frac{x}{n})^n$. We will later write $a_n(x)$ for a_n, when the dependence on x plays a role. Show that $a_{n+1}/a_n \geq 1$ whenever $x \geq 0$ or $n > |x| + 1$. (The previous lemma may help in estimating $[(1 + \frac{x}{n+1})/(1 + \frac{x}{n})]^n$.)

(b) Show that the sequence (a_n) is bounded above.
Hint: There may be a variety of ways; but you could use that $a_n(x)a_n(-x) < 1$ for $|x| < 1$ and a lower bound of $a_n(x)$ for $-1 < x < 0$ to get an upper bound for $0 < x < 1$. And maybe $a_n(x+y) < a_n(x)a_n(y)$ for $x > 0$.
(c) Defining $\exp(x) := \lim_{n\to\infty} a_n(x)$, prove that, for all $x, y \in \mathbb{R}$, it holds $\exp(x)\exp(-x) = 1$ and $\exp(x+y) = \exp(x)\exp(y)$. Also prove $x < y \Longrightarrow \exp(x) < \exp(y)$.

Solution: First note that the hypothesis $x \geq 0$ or $n \geq |x|+1$ guarantees that $1+\frac{x}{n}, 1+\frac{x}{n+1} > 0$. Now to **part (a):**

$$\frac{a_{n+1}}{a_n} = \left(\frac{1+\frac{x}{n+1}}{1+\frac{x}{n}}\right)^{n+1}\left(1+\frac{x}{n}\right) = \left(\frac{(n+1+x)n}{(n+x)(n+1)}\right)^{n+1}\left(1+\frac{x}{n}\right) = \left(1-\frac{x}{(n+x)(n+1)}\right)^{n+1}\left(1+\frac{x}{n}\right)$$

Using the lemma from #6 with $y = -x/(n+x)$ is permissible, because $-x/(n+x) > -1$. We conclude

$$\frac{a_{n+1}}{a_n} \geq \left(1-(n+1)\frac{x}{(n+x)(n+1)}\right)\left(1+\frac{x}{n}\right) = 1$$

Part (b): If $-1 < x \leq 0$, the sequence $(a_n(x))$ is trivially bounded above by 1, because $0 < 1+\frac{x}{n} < 1$. If $x \leq -1$, we can take $n_0 > |x|$ and still infer $0 < 1+\frac{x}{n} < 1$ for $n \geq n_0$. So the sequence is still bounded above by $\max\{a_1, a_2, \ldots, a_{n_0}, 1\}$. We have to show the boundedness for $x > 0$ yet.

We first assume $0 < x < 1$. Then $a_n(-x)a_n(x) = (1-x^2/n^2)^n < 1$. On the other hand $a_n(-x) \geq 1 - n\frac{x}{n} = 1-x > 0$ from Hwk #6. This guarantees $a_n(x) < \frac{1}{1-x}$.

Now let $x > 0$ be arbitrary and find $k \in \mathbb{N}$ such that $k > x$. Then we can use the previous estimate for $0 < x/k < 1$ and conclude.

$$a_n(x) = \left(1+\frac{x}{n}\right)^n = \left(1+k\frac{x/k}{n}\right)^n \leq_{\text{by \#6}} \left[\left(1+\frac{x/k}{n}\right)^k\right]^n \leq \left[\frac{1}{1-x/k}\right]^k$$

As a variant, let us note for $x, y \geq 0$ that $(1+\frac{x}{n})(1+\frac{y}{n}) = 1+\frac{x+y}{n}+\frac{xy}{n^2} \geq 1+\frac{x+y}{n}$. This implies $a_n(x+y) \leq a_n(x)a_n(y)$, and similar statements for an arbitrary number of summands by induction. (BTW, the same reasoning also applies for $x, y \leq 0$.) We again obtain the same conclusion via $a_n(x) = a_n(k \cdot x/k) \leq a_n(x/k)^k$.

Part (c): The limit $\lim_{n\to\infty} a_n(x)$ exists for each x, because the sequence is increasing from some $n = n_0$ on, and bounded above. Since for $n > |x|$, we can estimate

$$1 - n\cdot\frac{x^2}{n^2} \leq_{\text{by \#6}} \left(1-\frac{x^2}{n^2}\right)^n = a_n(x)a_n(-x) \leq 1\ ,$$

we conclude, by taking the limit, that

$$\lim_{n\to\infty} 1 - \frac{x^2}{n} \leq \exp(x)\exp(-x) \leq \lim_{n\to\infty} 1\ ,$$

hence $\exp(x)\exp(-x) = 1$.

We note that $a_n(x) > 0$ for every n sufficiently large (namely $n > |x|$) and also that (a_n) is increasing for sufficiently large n. This implies that $\exp(x) \geq a_n(x) > 0$.

We next conclude, for x, y either both nonnegative or both nonpositive, that $\exp(x+y) \leq \exp(x)\exp(y)$ because a similar inequality was proved in part (b) for the $a_n(x)$.

By taking reciprocals (which is allowed with reversal of the inequality, because both sides are positive), we conclude

$$\exp(-x-y) = \frac{1}{\exp(x+y)} \geq \frac{1}{\exp(x)\exp(y)} = \exp(-x)\exp(-y) \ .$$

But the opposite inequality is also true, since $-x$ and $-y$ again are either both nonnegative or both nonpositive.

So we have proved $\exp(x)\exp(y) = \exp(x+y)$ whenever x, y are both nonpositive or both nonnegative. If x and y have opposite signs, we note that $x+y$ has either the same sign as $-x$ or the same sign as $-y$ or is 0. Depending on which is the case, at least one of the following statements is valid:

$$\begin{array}{ll} \exp(x+y)\exp(-y) = \exp((x+y)+(-y)) = \exp(x) & \text{or} \\ \exp(x+y)\exp(-x) = \exp((x+y)+(-x)) = \exp(y) & \end{array}$$

But either of them implies $\exp(x+y) = \exp(x)\exp(y)$.

So we have proved the exponential law for arbitrary $x, y \in \mathbb{R}$.

Now $\exp(x) > 1$ for $x > 0$, because $1 < a_1(x)$ and $a_1(x) \leq \exp(x)$ from the monotonicity of the sequence. This, together with the exponential law, implies, for $x < y$ that $\exp(y) = \exp(y-x)\exp(x)$. Since $\exp(x), \exp(y) > 0$ we infer $\exp(y)/\exp(x) = \exp(y-x) > 1$, and therefore $\exp(y) > \exp(x)$.

Note: The following wasn't asked, but for $0 < x < 1$, we have $1 \leq \exp(x) \leq \frac{1}{1-x}$, and this implies (anticipating the definition of continuity 'officially' covered later in the course) that $\lim_{x\to 0+}\exp(x) = 1 = \exp(0)$. Similarly, $\lim_{x\to 0-}\exp(x) = 1 = \exp(0)$, so we prove the continuity of the exp function at 0. Finally, $\lim_{x\to x_0}\exp(x) = \lim_{x\to x_0}\exp(x-x_0)\exp(x_0) = \exp(0)\exp(x_0) = \exp(x_0)$ proves the continuity of exp at an arbitrary $x_0 \in \mathbb{R}$.

8: Let (a_n) be a sequence in $\mathbb{R}$ that is bounded below and 'sub-additive', i.e., $\forall n, m : a_{n+m} \leq a_n + a_m$. Prove that $\lim \frac{a_n}{n}$ exists. *Hints follow:*
(a) Prove for $k, n \in \mathbb{N}$ that $a_{kn} \leq k a_n$.
(b) Prove for $n, r \in \mathbb{N}$ that $\limsup_{k\to\infty} \frac{a_{kn+r}}{kn+r} \leq \frac{a_n}{n}$.
(c) You may assume the 'division with remainder theorem from elementary number theory': $\forall m, n \in \mathbb{N}\ \exists k, r \in \mathbb{N}_0 : m = kn + r$ and $0 \leq r \leq n-1$. Use it to prove $\limsup \frac{a_m}{m} \leq \inf \frac{a_n}{n}$ and conclude the original claim.

Solution:
Part (a) is an immediate induction over k. Trivially true for $k = 1$ (b/c $a_n \leq a_n$), the step from k to $k+1$ is

$$a_{(k+1)n} = a_{kn+n} \leq_{\text{subadd}} a_{kn} + a_n \leq_{\text{IH}} k a_n + a_n = (k+1)a_n \ .$$

Part (b): Since

$$\frac{a_{kn+r}}{kn+r} \leq \frac{a_{kn}+a_r}{kn+r} \leq \frac{k a_n + a_r}{kn+r}$$

we can take the $\limsup_{k\to\infty}$ on both sides (noticing that $b_k \leq c_k \implies \sup\{b_k : k \geq \ell\} \leq \sup\{c_k : k \geq \ell\} \implies \limsup b_k \leq \limsup c_k$) and conclude

$$\limsup_{k\to\infty} \frac{a_{kn+r}}{kn+r} \leq \limsup_{k\to\infty} \frac{k a_n + a_r}{kn+r}$$

The limsup on the right can actually be calculated as a limit since

$$\frac{ka_n + a_r}{kn + r} = a_n \frac{k}{kn + r} + \frac{a_r}{kn + r} = a_n \frac{1}{n + \frac{r}{k}} + \frac{a_r/k}{n + \frac{r}{k}} \ ,$$

which converges, as $k \to \infty$, to $a_n \frac{1}{n+r\cdot 0} + \frac{a_r \cdot 0}{n+r\cdot 0} = a_n/n$.

Part (c): Given $m \in \mathbb{N}$ and $n \in \mathbb{N}$, there are n cases to distinguish: Either there exists k such that $m = nk$, or there exists k such that $m = nk + 1$, or there exists k such that $m = nk + 2$, etc., the last case being $m = nk + (n - 1)$. This implies that

$$\Big\{\frac{a_m}{m} : m \geq \ell\Big\} = \bigcup_{r=0}^{n-1} \Big\{\frac{a_{nk+r}}{nk + r} : k \geq (\ell - r)/n\Big\} \subset \bigcup_{r=0}^{n-1} \Big\{\frac{a_{nk+r}}{nk + r} : k \geq \frac{\ell}{n} - 1\Big\}$$

Therefore

$$\sup\Big\{\frac{a_m}{m} : m \geq \ell\Big\} \leq \max_{r=0}^{n-1} \sup\Big\{\frac{a_{nk+r}}{nk + r} : k \geq \frac{\ell}{n} - 1\Big\}$$

(The right hand side is an upper bound for the union $\bigcup_r$, and therefore for the set on the left hand side. The sup on the left side is the smallest upper bound, hence $\leq$ the upper bound provided from the right side.)

As $\ell \to \infty$, so does $\frac{\ell}{n} - 1 \to \infty$, and we obtain (using the simple lemma: $\lim \max\{b_k, c_k\} = \max\{\lim b_k, \lim c_k\}$ provided the limits on the right exist – proof later):

$$\limsup \frac{a_m}{m} \leq \max_{r=0}^{n-1} \limsup_{k\to\infty} \frac{a_{kn+r}}{kn + r} \leq \max_{r=0}^{n-1} \frac{a_n}{n} = \frac{a_n}{n}$$

Since this inequality is true for every n, we can take the infimum (or $\liminf$) over all n and conclude $\limsup \frac{a_m}{m} \leq \liminf \frac{a_n}{n}$, from which the existence of the limit follows.

Since we never proved the lemma, let me prove it:

Lemma: *"Assume* $\lim b_k = b_*$ *and* $\lim c_k = c_*$*. Then* $\lim \max\{b_k, c_k\} = \max\{b_*, c_*\}$*."*

Proof: Without loss of generality assume $b_* \leq c_*$, so that $\max\{b_*, c_*\} = c_*$. Let $\varepsilon > 0$. Then there exists k_0 such that for all $k \geq k_0$ the following hold:

$$\begin{aligned} &b_* - \varepsilon < b_k < b_* + \varepsilon \leq c_* + \varepsilon \\ &c_* - \varepsilon < c_k < c_* + \varepsilon \end{aligned}$$

Since b_k and c_k are both $< c_* + \varepsilon$, so is their maximum: $\max\{b_k, c_k\} < c_* + \varepsilon$. On the other side, $\max\{b_k, c_k\} \geq c_k > c_* - \varepsilon$. Hence $\lim \max\{b_k, c_k\} = c_*$.

9: Using the dot product in $\mathbb{R}^n$, namely $\vec{u}\cdot\vec{v} := \sum_{i=1}^n u_i v_i$, we can write $\|\vec{u}\|_2^2 = \vec{u}\cdot\vec{u}$.
(a) Prove the Cauchy-Schwarz inequality $|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|_2\,\|\vec{v}\|_2$ by exploiting the fact that $(\vec{u}+t\vec{v})\cdot(\vec{u}+t\vec{v}) \ge 0$ for all $t\in\mathbb{R}$ and then choosing a special t. (Actually the one that, according to Calculus 1, minimizes the given expression. But note that, logically, you do not need to justify your choice of t.)
(b) Use it to conclude $\|\vec{u}+\vec{v}\|_2 \le \|\vec{u}\|_2 + \|\vec{v}\|_2$.
[too elementary: I won't collect this one for grading.]

Solution: It is obvious from the definition that the distributive law applies for the dot product with addition; dito the rule $\vec{u}\cdot(t\vec{v}) = t(\vec{u}\cdot\vec{v})$ and the commutative law. So we have

$$0 \le (\vec{u}+t\vec{v})\cdot(\vec{u}+t\vec{v}) = \vec{u}\cdot\vec{u} + 2t\vec{u}\cdot\vec{v} + t^2\vec{v}\cdot\vec{v} = \|\vec{u}\|_2^2 + 2t\vec{u}\cdot\vec{v} + t^2\|\vec{v}\|_2^2$$

Since the Cauchy-Schwarz inequality is trivially true when $\vec{v}=0$, we may now assume $\vec{v}\neq 0$. Choosing $t = -(\vec{u}\cdot\vec{v})/\|\vec{v}\|_2^2$ we obtain

$$0 \le \|\vec{u}\|_2^2 - 2\frac{(\vec{u}\cdot\vec{v})^2}{\|\vec{v}\|_2^2} + \frac{(\vec{u}\cdot\vec{v})^2}{\|\vec{v}\|_2^2}$$

and hence by simplifying $(\vec{y}\cdot\vec{v})^2 \le (\|\vec{u}\|_2\,\|\vec{v}\|_2)^2$. Taking the square root yields the CSI.

10: Prove the triangle inequality for $\|\cdot\|_\infty$ on $\mathcal{BF}(X\to\mathbb{R}) = \{f : X\to\mathbb{R} : f \text{ bounded}\}$. Here $\|f\|_\infty := \sup\{|f(x)| : x\in X\}$ and X is any set.

Solution: We need to show $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ for any functions $f,g\in\mathcal{BF}(X\to\mathbb{R})$. First note that for any $x\in X$, we have $|(f+g)(x)| := |f(x)+g(x)| \le |f(x)|+|g(x)| \le \|f\|_\infty + \|g\|_\infty$.

Taking the supremum over $x\in X$ on the left immediately gives the claim.

11: I mentioned the norms $\|u\|_p := (\sum_{i=1}^n |u_i|^p)^{1/p}$ without proof of the norm properties, where $p\ge 1$. Now try $p=\frac{1}{2}$. Is it a norm? Proof or counterexample.

Solution: The function $\|\cdot\|_{1/2}$ given by the formula above is *not* a norm, despite the appearance created by misusing the symbol $\|\cdot\|$. While the homogeneity and positivity properties of the norm are indeed verified (trivially), the triangle inequality is violated.

Here is a counterexample in the case $n=2$: Let $u=(1,0)$ and $v=(0,1)$. Then

$$\|u+v\|_{1/2} = \|(1,1)\|_{1/2} = (\sqrt{1}+\sqrt{1})^2 = 4 \quad\text{whereas}\quad \|u\|_{1/2}+\|v\|_{1/2} = 1+1 = 2$$

For larger n, the same example, padded with 0's in the extra components, can be used.

Of course the trivial case $n=1$ is an exception: there, any $\|u\|_p$ (with $p>0$) is equal to $|u|$ and is therefore a norm.

12: Assume d is a distance function on X. Show that $d_1 := \frac{d}{1+d}$ is also a distance function. Show that a sequence is convergent with respect to d if and only if it is convergent with respect to d_1.

Solution: $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$ is obvious from $d(x,y) \geq 0$. Also $d_1(x,y) = 0$ if and only if $d(x,y) = 0$. Likewise, symmetry of d_1 follows trivially from symmetry of d. The only nontrivial issue is to verify the triangle inequality:

Let $x, y, z \in X$ and call $d(x,y) =: a$, $d(y,z) =: b$, and $d(x,z) =: c$. We know from the triangle inequality for d that $c \leq a+b$. And we also have $a, b, c \geq 0$. We have to show $\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}$.

Since the function $c \mapsto \frac{c}{1+c} = 1 - \frac{1}{1+c}$ is increasing ($c < c' \implies \frac{c}{1+c} < \frac{c'}{1+c'}$), it suffices to show the second step in the following inequality:

$$\frac{c}{1+c} \leq \frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}$$

Multiplying with the common denominator (which is positive), this last claimed inequality is equivalent to

$$(a+b)(1+a)(1+b) \leq a(1+a+b)(1+b) + b(1+a+b)(1+a)$$

Subtracting $(a+b)(1+a+b)$, this is equivalent to

$$(a+b)ab \leq (1+a+b)\, 2ab\,,$$

which is clearly true since $a+b \leq 1+a+b$ and $ab \leq 2ab$. Since the steps form one inequality to the next were equiavlences, this proves the triangle inequality for d_1.

Now since $d_1 \leq d$, we argue that if $d(x_n, x) < \varepsilon$ then $d_1(x_n, x) < \varepsilon$. Dittoing all the quantifiers in the definition, we see: If (x_n) is Cauchy wrt d, then it is Cauchy wrt d_1; and if $\lim x_n = x$ wrt d, then $\lim x_n = x$ wrt d_1.

For the converse direction, we notice that $d = d_1/(1-d_1)$ and argue: If $d_1 < \varepsilon$ and $\varepsilon \leq \frac{1}{2}$ then $d < 2\varepsilon$. Suppose (x_n) is Cauchy wrt d_1 and let $\varepsilon > 0$. Using the definition of Cauchy with $\varepsilon' = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$, we get an n_0 such that for $n, m \geq n_0$, it holds $d_1(x_n, x_m) < \varepsilon' \leq \frac{1}{2}$. This implies $d(x,y) < 2\varepsilon' \leq \varepsilon$. So (x_n) is Cauchy wrt d. The same reasoning applies to convergence.

13: On $\mathbb{N}$ a distance function d_1 is given by $d_1(n,m) := |n-m|$. Another distance function is given by $d_2(n,m) := |\frac{1}{n} - \frac{1}{m}|$. Which are the convergent sequences in each case? Which are Cauchy sequences in each case?

Solution: We call a sequence (x_n) 'eventually constant' if there exists an n_0 such that for all $n, m \geq n_0$, it holds $x_n = x_m$. We say that a sequence x_n in $\mathbb{N}$ goes to infinity if $\liminf x_n = \infty$ as defined within $\mathbb{R}$; in other words if for every $M > 0$, there exists a $k \in N$ such that $a_n \geq M$ for $n \geq k$.

We claim:
(1) A sequence in $\mathbb{N}$ is Cauchy wrt the usual metric d_1 if and only if it is eventually constant, and also, if an only if it converges.
(2) A sequence in $\mathbb{N}$ converges wrt d_2 if and only it it is eventually constant. It is Cauchy if and only if it is eventually constant or goes to infinity.

To prove this we only have to show:
(a) An eventually constant sequence converges wrt any metric (with the limit being that same constant).
(b) A Cauchy sequence wrt d_1 is eventually constant.

(c) A Cauchy sequence wrt d_2 is eventually constant or goes to infinity.
(d) If a sequence goes to infinity, then it is Cauchy wrt d_2.
(e) If a sequence converges wrt d_2, it is eventually constant.

Then (1) follows from concatenating: Cauchy $\Longrightarrow_{(b)}$ ev.const. $\Longrightarrow_{(a)}$ convergent $\Longrightarrow_{\text{general Thm}}$ Cauchy.

Also (2) follows similarly: Cauchy $\Longrightarrow_{(b)}$ ev.const. or to-infinity $\Longrightarrow_{\text{like (1), or (d)}}$ Cauchy. And Convergent $_{(a)}\Longleftrightarrow_{(e)}$ ev.const.

Now (a) is trivial: If for all $n, m \geq n_0$ we have $x_n = x_m$, then we call this common value x_*, and have $d(x_n, x_*) = 0 < \varepsilon$. The same n_0 from the def. of eventually constant works for all $\varepsilon > 0$ in the def. of convergence.

For (b), use the Cauchy property with $\varepsilon = \frac{1}{2}$ to conclude: $\exists n_0\ \forall n, m \geq n_0 : d(x_n, x_m) < \frac{1}{2}$. Since absolute values of differences between natural numbers are always natural numbers or 0, this implies $d_1(x_n, x_m) = 0$, hence $x_n = x_m$. So Cauchy implies eventually constant.

For (c), we assume (x_n) is Cauchy and does not go to infinity. The second property means $\exists M\ \forall k \in \mathbb{N}\ \exists n \geq k : x_n \leq M$. It is no loss of generality to assume $M \geq 1$. Now we use the definition of Cauchy with $\varepsilon = \frac{1}{2M(M+1)}$. Then $d_2(x_n, x_m) < \varepsilon$ means $|\frac{1}{x_n} - \frac{1}{x_m}| < \frac{1}{2M(M+1)}$, which is equivalent to

$$\frac{|x_n - x_m|}{x_n\, x_m} < \frac{1}{2M(M+1)}$$

This has to hold for every $n, m \geq n_0$, in particular we choose n in such a manner that $x_n \leq M$. Then $x_m \leq x_n + |x_n - x_m| < x_n + \varepsilon < x_n + 1$ and we conclude $|x_n - x_m| < \frac{x_n(x_n+1)}{2M(M+11)} \leq \frac{1}{2}$. Since x_n and x_m are natural numbers, this implies $x_n = x_m$. So we have concluded that (x_n) is eventually constant.

As for (d), if (x_n) goes to infinity, we assume $\varepsilon > 0$ and choose $M > \frac{2}{\varepsilon}$. Then there is a k such that for any $n, m \geq k$, we have $x_n, x_m \geq M$, hence $\frac{1}{x_n}, \frac{1}{x_m} < \frac{\varepsilon}{2}$. But then $d_2(x_m, x_n) = |\frac{1}{x_n} - \frac{1}{x_m}| \leq \frac{1}{x_n} + \frac{1}{x_m} < \varepsilon$. So (x_n) is Cauchy.

For (e) we assume $\lim x_m = x_*$ wrt d_2. We copy the proof of (c) with x_* replacing both x_n and M.

Comment: This example shows that even if two metrics on the same set define the same notions of convergence (and subsequently the same notions of open sets, as we will see for this example), they may still define different notions of Cauchy, in other words, the property of completeness is not determined by the topology alone, but depends on the metric.

14: The following example is instructive, albeit not significant for purposes of Calculus: You may and should use basic facts about prime factor decompositions of rational numbers in this example. Given a rational $q = \frac{m}{n} \in \mathbb{Q}$ with $m \in \mathbb{Z}, n \in \mathbb{N}$, we consider the prime factor decomposition $q = \pm p_1^{k_1} \cdots p_r^{k_r}$. Choosing a particular prime p, we define $[0]_p := 0$; and for $q \neq 0$, we define $[q]_p := 1/p^k$, where p^k is the power of p that occurs in the prime factor decomposition of q. *Example:* $[\frac{12}{25}]_3 = \frac{1}{3}$, $[\frac{2}{9}]_3 = 9$, $[\frac{2}{5}]_3 = [1]_3 = 1$.

Prove the strengthened triangle inequality $[x + y]_p \leq \max\{[x]_p, [y]_p\}$ and show that $d_p(x, y) := [x - y]_p$ defines a metric on $\mathbb{Q}$. (It is called the p-adic metric).

Let $a_n := \frac{2^n}{2^n+1}$. Find the limit $\lim a_n$ in $\mathbb{Q}$ with the metric $d(x, y) := |x - y|$ (that is too easy). Next find $\lim a_n$ with respect to the metric d_2.

Solution: We let p be any chosen prime number. Every rational number x can be written in the form $x = \pm p^k \dfrac{a}{b}$ where $k \in \mathbb{Z}$ and a and b are integers that do not contain p in their prime factorization. (We may assume that the prime factors in a and in b are distinct, i.e., that $\frac{a}{b}$ is in lowest terms, but we will not need this.) Likewise, we write $y = \pm p^\ell \dfrac{c}{d}$ with the analogous hypotheses.

We now prove the strengthened triangle inequality for x and y, applying the notation just established. Without loss of generality, we assume $k \geq \ell$.

Then $[x]_p = p^{-k}$ and $[y]_p = p^{-\ell}$ and $\max\{[x]_p, [y]_p\} = p^{-\ell}$. To calculate $[x+y]_p$ we write

$$x + y = \pm p^k \frac{a}{b} \pm p^\ell \frac{c}{d} = \pm p^\ell \, \frac{p^{k-\ell}ad \pm bc}{bd} \; .$$

Now the denominator, the integer bd, does not contain a prime p because neither b nor d does. The numerator $p^{k-\ell}ad \pm bc$ is an integer and it may or may not contain a prime p. If it does not, then $[x+y]_p = p^{-\ell}$ (so it exactly equals the maximum in question); on the other hand if the numerator does contain p (to some positive power s, exactly), then $[x+y]_p = p^{-\ell-s}$ and thus smaller than the maximum in question.

This proves the strengthened triangle inequality. We can actually say more: If $k = \ell$, i.e., $[x]_p = [y]_p$, then strict inequality may indeed hold (but doesn't need to). However if $k > \ell$, i.e., $[x]_p \neq [y]_p$, then the numerator will not contain an extra factor p and the strengthened triangle inequality becomes an equality. This is because if p divided the sum $p^{k-\ell}ad \pm bc$ in the numerator and is known to divide one summand $p^{k-\ell}ad$, then it would have to divide the other summand bc also, which is however ruled out since neither b nor c has a factor p.

So we have actually proved:
$[x+y]_p \leq \max\{[x]_p, [y]_p\}$ **and** if $[x]_p \neq [y]_p$ then equality holds. $(*)$

Now for the specific sequence $a_n = \frac{2_n}{2^n+1}$. In the usual metric d, $\lim a_n = 1$ because

$$|a_n - 1| = \frac{1}{2^n+1} < \frac{1}{2^n} < \frac{1}{n} \, ,$$

and so $\lim a_n = 0$ b/c $\lim \frac{1}{n} = 0$. We have used the lemma "$\forall n \in \mathbb{N} : 2^n > n$" in this calculation. It is very easy to prove this lemma by induction.

In contrast, wrt to the p-adic metric d_p for $p = 2$, we have $\lim x_n = 0$ because $[a_n]_2 = 2^{-n} < \frac{1}{n}$ (since the denominator of a_n, namely $2^n + 1$, does not contain a prime factor 2).

Note: If we were to study the sequence wrt another p-adic metric, say we choose $p = 3$, then it is easy to see that 3 divides $2^n + 1$ for odd n, but does not divide $2^n + 1$ for even n. So we know $[a_n]_3 = 1$ for even n and $[a_n]_3 \geq 3^1$ for odd n. The amended version $(*)$ then tells us that $d_3(a_m, a_n) \geq 3$ whenever one of m, n is even and the other is odd. In particular (a_n) is *not* a Cauchy sequence wrt d_3.

If you show, by induction over k, that 3^{k+1} divides $2^{3^k+1} + 1$, you can even conclude that $[a_n]_3$ is unbounded.

So we see why this example is not from the analyst's playground, but more from algebra and number theory. However, ultrametrics, which are metrics that satisfy the strengthened triangle inequality with max instead of +, may occur occasionally in contexts of interest to us.

15: Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that does not have a fixed point, but satisfies a weakened contraction condition $|f(x) - f(y)| < |x - y|$ whenever $x \neq y$. For this example, you may use Calculus 1 knowledge (even though I think it can also be done with the material constructed in this class already.)

Solution: We could for instance take $f(x) = \frac{1}{2}(x + \sqrt{x^2 + 1})$. Clearly $f(x) > \frac{1}{2}(x + |x|) \geq x$ so f has no fixed point.

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y| + \frac{1}{2}|\sqrt{x^2 + 1} - \sqrt{y^2 + 1}| = \frac{1}{2}|x - y| + \frac{1}{2}\frac{|x^2 - y^2|}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}}$$

So

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \frac{1}{2} + \frac{1}{2}\frac{|x + y|}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}} < \frac{1}{2} + \frac{1}{2}\frac{|x + y|}{|x| + |y|} \leq 1$$

Variants: If we are willing to rely on claculus 1 knowledge, we are looking for a function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is everywhere < 1 and > -1, and which is always $> x$ or else always $< x$. The derivative guarantees by the mean value theorem that $|f(x_2) - f(x_1)| < |x_2 - x_1|$ whenever $x_2 \neq x_1$. Examples of such functions are given by the formulas $\ln(1 + e^x)$, or $x - \arctan x - \frac{\pi}{2}$.

16: Prove that $f : [0, \infty[\to \mathbb{R}, \ x \mapsto \sqrt{x}$ is uniformly continuous. (Of course continuity is understood here with respect to the usual distance on $\mathbb{R}$.)

Solution:
(1) The following 'routine' solution shows continuity, but fails to prove uniform continuity; it is just offered for insight of instruction; only (2) below is needed:

For continuity at $x_0 = 0$, we assume $\varepsilon > 0$ is given and choose $\delta = \varepsilon^2$. Then we estimate, for $x \geq 0$ and $|x - x_0| < \delta$ that

$$|\sqrt{x} - \sqrt{x_0}| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

For continuity at $x_0 > 0$, we assume $\varepsilon > 0$ given and choose $\delta = \sqrt{x_0}\,\varepsilon$ to estimate for $|x - x_0| < \delta$ and $x \geq 0$ that

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{\delta}{\sqrt{x_0}} = \varepsilon$$

So the square root function is continuous at every $x_0 \geq 0$.

(2) A somewhat smarter proof proves uniform continuity, making (1) obsolete: Given $\varepsilon > 0$, we choose $\delta = \varepsilon^2$ (motivated by the intuitive insight that $x_0 = 0$ should be the 'worst case scenario'). We assume $x, y \geq 0$ with $|x - y| < \delta$ and want to estimate $|\sqrt{x} - \sqrt{y}|$ to show it is $< \varepsilon$. To this end, assume w/olog that $y \geq x$. Two different estimates will be used depending on the size of y: If $y < \delta$, we argue

$$|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x} \leq \sqrt{y} < \sqrt{\delta} = \varepsilon$$

If $y \geq \delta$, we estimate

$$|\sqrt{y} - \sqrt{x}| = \frac{|y - x|}{\sqrt{y} + \sqrt{x}} < \frac{\delta}{\sqrt{y}} \leq \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} = \varepsilon$$

17: Let (X, d) be the metric space of $\mathbb{R}$ with the *discrete* metric $d(x, y) := 1$ whenever $x \neq y$. Let (Y, d) be *any* metric space, and $\mathbb{R}$ be equipped with the usual metric.

(a) Show that every function $f : X \to Y$ is continuous. Is the same true with 'uniformly continuous'?
(b) Conjecture which functions $f : \mathbb{R} \to X$ are continuous (we'll prove it in class).

Solution:
(a) to show continuity of $f : X \to Y$ where X is a discrete metric space, given $\varepsilon > 0$, we choose $\delta = \frac{1}{2}$. Then the claim "if $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \varepsilon$" is true, because the hypothesis $d(x_1, x_2) < \frac{1}{2}$ implies $x_1 = x_2$ in a discrete metric space. Then $f(x_1) = f(x_2)$, so $d(f(x_1), f(x_2)) = 0 < \varepsilon$. So if the domain is a discrete metric space, then every f is uniformly continuous.

(b) We claim that the only continuous functions from $\mathbb{R}$ to a discrete metric space are the constant functions. It is clear that constant functions are continuous, we only have to prove the implication "continuous $\Longrightarrow$ constant".

So let $f : \mathbb{R} \to X$ be continuous at each x_0 and choose $\varepsilon = \frac{1}{2}$ in the definition. Then there exists $\delta_0 > 0$ such that for $|x - x_0| < \delta_0$, we have $d(f(x), f(x_0)) < \varepsilon = \frac{1}{2}$, i.e $f(x) = f(x_0)$. We first use this argument for *one* x_0, (we could take $x_0 = 0$) and conclude there exists a value $C = f(x_0)$ and a $\delta_0 > 0$ such that $f(x) = C$ for $x_0 - \delta_0 < x < x_0 + \delta_0$. Then we define $S := \{\delta > 0 \mid \forall x \in [x_0, x_0 + \delta[; f(x) = C\}$. We have just shown that $S \neq \emptyset$. We next want to show (by indirect argument) that S is unbounded above; this will then immediately imply that $f(x) = C$ for all $x > x_0$.

So let's assume that S is bounded above; define $s := \sup S$ and let $x_1 := x_0 + s$. Since f is continuous at x_1, we know that there exists $\delta_1 > 0$ (and w/olog $\delta_1 < s$) such that for all $x \in]x_1 - \delta_1, x_1 + \delta_1[$, it holds $f(x) = f(x_1)$. In particular, choosing $x = x_1 - \delta_1/2$, we have $f(x) = C$, hence $f(x_1) = C$. But this means that $f(x) = C$ for $x \in [x_0, x_1[\cup]x_1 - \delta_1, x_1 + \delta_1[= [x_0, x_0 + s + \delta_1[$. So $s + \delta_1 \in S$, and this contradicts the fact that s was the supremum of S.

This contradiction shows $f(x) = C$ for all $x \geq x_0$. A similar argument shows $f(x) = C$ for all $x \leq 0$. So f is constant.

18: Suppose (X, d_X) and (Y, d_Y) are metric spaces: the set $X \times Y$ can be made into a metric space by defining $d_\infty((x, y), (x', y')) := \max\{d_X(x, x'), d_Y(y, y')\}$.
(a) Prove this.
(b) With this definition of the metric on $X \times X$, write out what the statement "$d_X : X \times X \to \mathbb{R}$ is continuous" means and then prove it. Decide if the same statement is true with 'uniformly continuous' instead of merely 'continuous'.

Solution:
(a) Symmetry and positivity are easy:
$d_\infty((x, y), (x', y')) = d_\infty((x', y'), (x, y))$, because the symmetry for d_X and d_Y gives $\max\{d_X(x, x'), d_Y(y, y')\} = \max\{d_X(x', x), d_Y(y', y)\}$.

$d_\infty((x, y), (x', y')) \geq 0$ because $d_X(x, x') \geq 0$ (or because $d_Y(y, y') \geq 0$). If $d_\infty((x, y), (x', y')) = 0$, then both $d_X(x, x')$ and $d_Y(y, y')$ are ≤ 0 (and always ≥ 0, hence 0), so $x = x'$ and $y = y'$, implying $(x, y) = (x', y')$.

The triangle inequality is proved as in #10: We have to show
$d_\infty((x, y), (x'', y'')) \leq d_\infty((x, y), (x', y')) + d_\infty((x', y'), (x'', y''))$, i.e.,
$\max\{d_X(x, x''), d_Y(y, y'')\} \leq \max\{d_X(x, x'), d_Y(y, y')\} + \max\{d_X(x', x''), d_Y(y', y'')\}$.

We have
$d_X(x,x'') \le d_X(x,x') + d_X(x',x'') \le \max\{d_X(x,x'), d_Y(y,y')\} + \max\{d_X(x',x''), d_Y(y',y'')\}$
and likewise
$d_Y(y,y'') \le d_Y(y,y') + d_Y(y',y'') \le \max\{d_X(x,x'), d_Y(y,y')\} + \max\{d_X(x',x''), d_Y(y',y'')\}$.
Therefore the max of the two left hand sides is less or equal the common right hand side.

(b) The statement "$d_X : X \times X \to \mathbb{R}$ is continuous at (x,y)" is defined to be:

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall (x',y') : d_\infty((x,y),(x',y')) < \delta \Longrightarrow |d_X(x,y) - d_X(x',y')| < \varepsilon\ ,$$

or, using the definition of d_∞,

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall (x',y') : d_X(x,x') < \delta \wedge d_X(y,y') < \delta \Longrightarrow |d_X(x,y) - d_X(x',y')| < \varepsilon\ .$$

To prove this statement, we use the triangle inequality for d_X and say

$$\begin{aligned} &d_X(x,y) \le d_X(x,x') + d_X(x',y') + d_X(y',y) \qquad \text{hence} \\ &d_X(x,y) - d_X(x',y') \le d_X(x,x') + d_X(y',y)\ . \end{aligned}$$

Given $\varepsilon > 0$, we now choose $\delta := \frac{1}{2}\varepsilon$ and conclude that the right hand side is $< \delta + \delta = \varepsilon$. By swapping the primed with the unprimed variables, we also prove $d_X(x',y') - d_X(x,y) < \varepsilon$ with the same choice of δ.

This proves the continuity of d. Actually, we have proved uniform continuity because the choice of δ did not depend on (x,y).

19: Let $g : X \to Y$ and $f : Y \to Z$ be functions, where X, Y, Z are metric spaces; let $f \circ g : X \to Z\ ,\ x \mapsto f(g(x))$ be their composition. If g is continuous at x_0 and f is continuous at $y_0 := g(x_0)$, show that $f \circ g$ is continuous at x_0.

Solution: Let $x_0 \in X$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that $d(x,x_0) < \delta$ implies $d(\,f(g(x)), f(g(x_0))\,) < \varepsilon$.

Since f is continuous at $y_0 := g(x_0)$, there is an $\eta > 0$ such that $d(y,y_0) < \eta$ implies $d(f(y), f(y_0)) < \varepsilon$. We take this η as the ε in the definition of continuity for g. Namely, there exists $\delta > 0$ such that $d(x,x_0) < \delta$ implies $d(g(x), g(x_0)) = d(g(x), y_0) < \eta$. Using the continuity estimate for f with $y := g(x)$ we therefore obtain $d(\,f(g(x)), f(y_0)\,) < \varepsilon$ as required.

20: Assume X and Y are metric spaces, where the metric on Y is bounded. Per #12, boundedness of the metric is no loss of generality. We can define $\mathcal{F}(X \to Y)$ to be the set of all functions from X to Y and make it into a metric space by defining $d_\infty(f,g) := \sup\{d(f(x), g(x)) : x \in X\}$. (You don't need to prove these simple facts; they are similar to #10.)

Show that convergence $f_n \to f$ in the sense of the metric d_∞ is equivalent to uniform convergence $f_n \to f$.

We use the metric d_∞ on $C^0(X \to Y)$, the subset of *continuous* functions from X to Y. Show this metric space is complete, if Y is complete.

Solution: Let me note beforehand that the assumption that the metric on Y be bounded is only used to guarantee that the sup by which d_∞ is defined is a real number (rather than the symbol ∞), but will not be needed anywhere below.

Convergence in the sense of the metric means (with the definition of d_∞ written out):

$$(MC) \qquad \forall \varepsilon > 0\, \exists n_0\, \forall n \geq n_0 : \sup\{d(f_n(x), f(x)) : x \in X\} < \varepsilon$$

Uniform convergence means:

$$(UC) \qquad \forall \varepsilon > 0\, \exists n_0\, \forall n \geq n_0\, \forall x \in X : d(f_n(x), f(x)) < \varepsilon$$

Now clearly (MC) implies (UC), with the same choice of n_0, since $d(\ldots) \leq \sup\{d(\ldots)\} < \varepsilon$. Conversely, if (UC) holds, we may use it with 0.9ε instead of ε, and when $d(f_n(x), f(x)) < 0.9\varepsilon$ for all $x \in X$, then the supremum over these x is at least $\leq 0.9\varepsilon < \varepsilon$, hence we conclude (MC).

Now to prove completeness of $C^0(X \to Y)$, suppose we have a Cauchy sequence (f_n) in this space, i.e., again writing out the definition of d_∞,

$$(MCS) \qquad \forall \varepsilon > 0\, \exists n_0\, \forall n, m \geq n_0 : \sup\{d(f_n(x), f_m(x)) : x \in X\} < \varepsilon$$

Again, since for each x, we have $d(\ldots) \leq \sup\{d(\ldots)\}$, we infer that for each $x \in X$, the sequence $(f_n(x))_n$ is a Cauchy sequence in Y. The completness of Y guarantees us that this Cauchy sequence has a limit, and we define $f(x) := \lim f_n(x)$.

We have thus constructed a function $f : X \to Y$ as a pointwise limit of the f_n. (At this moment, we wouldn't know yet whether f is actually continuous.) Now we will show that $f_n \to f$ *uniformly.* From (MCS), we get:

$$\forall \varepsilon > 0\, \exists n_0\, \forall n, m \geq n_0\, \forall x \in X : d(f_n(x), f_m(x)) < \varepsilon\ .$$

Taking the limit $m \to \infty$ in $d(f_n(x), f_m(x)) < \varepsilon$ (and using the continuity of d), we obtain $d(f_n(x), f(x)) \leq \varepsilon$, i.e., $f_n \to f$ uniformly (except that we have $\leq \varepsilon$ instead of $< \varepsilon$, an inconsequential difference, since we can use 0.9ε instead of ε again).

By the theorem in class, uniform limits of continuous functions are continuous, so $f \in C^0(X \to Y)$. And by the equivalence of metric convergence with uniform convergence, we know $f_n \to f$ in the metric sense. This proves completeness of (C^0, d_∞).

Note: As an aside, there does not exist a metric on $C^0([0,1] \to \mathbb{R})$ for which convergence in the sense of the metric is equivalent to pointwise convergence. A proof can be found in "M.K. Fort: A note on pointwise convergence, Proceedings of the AMS, **2** (1951), pp. 34–35".

21: (a) Prove that $\exp : \mathbb{R} \to \mathbb{R}_+$ as defined previously is bijective.
(b) We'll denote its inverse function as ln, so $y = \ln x :\iff x = \exp(y)$. Prove that $\ln : \mathbb{R}_+ \to \mathbb{R}$ is increasing and continuous. Also prove $\ln(xy) = \ln x + \ln y$.
(c) For $x > 0$ and $q \in \mathbb{R}$, we define $x^q := \exp(q \ln x)$. Prove for all $x > 0$ and $q_1, q_2 \in \mathbb{R}$ that $x^{q_1+q_2} = x^{q_1} x^{q_2}$ and $x^{q_1 q_2} = (x^{q_1})^{q_2}$. Also prove $(xy)^q = x^q y^q$ for all $x, y > 0$ and $q \in \mathbb{R}$. Note that $(x, q) \mapsto x^q$, $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous.
(d) letting $e := \exp(1)$, show that $\exp(x) = e^x$

Solution: (a) We know from #7 that values of exp are positive since they are $\geq a_n(x) = (1 + \frac{x}{n})^n > 0$ provided $n > |x|$. We also know that $x_1 < x_2$ implies $\exp(x_1) < \exp(x_2)$ (and likewise with $>$), so by trichotomy and contrapositive, $\exp(x_1) = \exp(x_2)$ implies $x_1 = x_2$.

We have seen in class that exp is continuous. To show that any $y > 0$ is in the range of exp, we first assume $y \geq 1$. Then $\exp(y) \geq 1 + y > y$, whereas $\exp(0) = 1 \leq y$. By the intermediate value theorem there exists $x \in [0, y]$ for which $\exp(x) = y$. If in contrast $y < 1$, but still > 0, then $1/y > 1$ and therefore there exists x such that $\exp(x) = 1/y$. But then $\exp(-x) = y$. So we have shown that exp is *onto* $\mathbb{R}_+$.

(b) Suppose $y_1 < y_2$. We want to show $\ln y_1 < \ln y_2$. By contrapositive, we have to show: If $\ln y_1 \geq \ln y_2$, then $y_1 \geq y_2$. This indeed follows from the fact that exp is increasing and $\exp \ln y = y$.

To show continuity of ln at y_0, we have to show:

$$\forall \varepsilon > 0\, \exists \delta > 0 : |y - y_0| < \delta \Longrightarrow |\ln y - \ln y_0| < \varepsilon \,.$$

We show this by showing the contrapositive (and let $x := \ln y$, $x_0 := \ln y_0$); so we show:

$$\forall \varepsilon > 0\, \exists \delta > 0 : |x - x_0| \geq \varepsilon \Longrightarrow |\exp x - \exp x_0| \geq \delta \,.$$

We do this by letting $\delta := \frac{\varepsilon}{1+\varepsilon} \exp x_0$. Then, assuming $|x - x_0| \geq \varepsilon$, we have either $x \geq x_0 + \varepsilon$ or $x \leq x_0 - \varepsilon$. If $x \geq x_0 + \varepsilon$, then

$$\begin{aligned} \exp(x) &\geq \exp(x_0 + \varepsilon) = \exp(x_0) \exp(\varepsilon) \geq \exp(x_0)\,(1 + \varepsilon) = \\ &= \exp(x_0) + \varepsilon \exp(x_0) = \exp(x_0) + (1 + \varepsilon)\delta > \exp(x_0) + \delta \end{aligned}$$

On the other hand, if $x \leq x_0 - \varepsilon$, then

$$\begin{aligned} \exp(x) &\leq \exp(x_0 - \varepsilon) = \exp(x_0) \exp(-\varepsilon) \leq \exp(x_0)\, \tfrac{1}{1+\varepsilon} = \\ &= \exp(x_0) - \tfrac{\varepsilon}{1+\varepsilon} \exp(x_0) = \exp(x_0) - \delta \end{aligned}$$

This proves continuity of ln.

Since exp is bijective, the claim $\ln(xy) = \ln x + \ln y$ is equivalent to $\exp(\ln(xy)) = \exp(\ln x + \ln y)$. But the left hand side is xy, the right hand side is $\exp(\ln x)\, \exp(\ln y) = xy$ as well.

(c)

$$x^{q_1 + q_2} := \exp((q_1 + q_2) \ln x) = \exp(q_1 \ln x + q_2 \ln x) = \exp(q_1 \ln x)\, \exp(q_2 \ln x) = x^{q_1}\, x^{q_2}$$

$$(x^{q_1})^{q_2} = \exp[q_2 \ln(x^{q_1})] = \exp[q_2 \ln(\exp(q_1 \ln x))] = \exp[q_2\, q_1 \ln x] = x^{q_2\, q_1} = x^{q_1\, q_2}$$

$$(xy)^q = \exp[q \ln(xy)] = \exp[q(\ln x + \ln y)] = \exp[q \ln x + q \ln y] = \exp(q \ln x)\, \exp(q \ln y) = x^q\, y^q$$

The power function is continuous, because it is a composition of the ln function (more precisely (id, ln)), the product function, and the exp function, all of which are continuous. More precisely:

$$\begin{array}{ccccccc} (q, x) & \mapsto & (q, \ln x) & \mapsto & q \cdot \ln x & \mapsto & \exp(q \ln x) \\ \Cap & (\mathrm{id}, \ln) & \Cap & \text{times} & \Cap & \exp & \Cap \\ \mathbb{R} \times \mathbb{R}_+ & \to & \mathbb{R} \times \mathbb{R} & \to & \mathbb{R} & \to & \mathbb{R}_+ \end{array}$$

(d) $e^x = \exp(x \ln e) = \exp(x \ln(\exp 1)) = \exp(x \cdot 1) = \exp(x)$.

22: Having constructed arbitrary powers, it's time to introduce the inequality of the arithmetic and geometric mean (short: agm inequality): "If $n \in \mathbb{N}$ and

$x_1, x_2, \ldots, x_n \in \mathbb{R}_+$, then $(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$. The inequality is strict unless $x_1 = \ldots = x_n$."

(a) Prove the inequality for $n = 2^k$ by induction over k.

(b) Then prove it for arbitrary n by choosing some k such that $m := 2^k > n$ and defining $x_{n+1}, \ldots, x_m$ wisely.

(I won't collect this for grading, but if you're not familiar with it, you should do it, as it is a 'must know'.)

Solution: (a) Start the induction with $k = 1$, i.e., $n = 2$. We need to show $\sqrt{x_1 x_2} \leq \frac{1}{2}(x_1 + x_2)$. Since both sides are positive, this is equivalent to $x_1 x_2 \leq \frac{1}{4}(x_1 + x_2)^2$, or equivalently, $4x_1 x_2 \leq x_1^2 + 2x_1 x_2 + x_2^2$. This in turn is equivalent to $0 \leq x_1^2 - 2x_1 x_2 + x_2^2$, which is true, because the right hand side is a square, $(x_1 - x_2)^2$. Clearly, equality holds only when $x_1 - x_2 = 0$.

Now the induction step from k to $k + 1$, i.e., from n to $2n$:

$$(x_1 x_2 \ldots x_{2n})^{1/(2n)} = \sqrt{(x_1 \ldots x_n)^{1/n} (x_{n+1} \ldots x_{2n})^{1/n}} \,. \qquad (*)$$

By induction hypothesis,

$$(x_1 \ldots x_n)^{1/n} \leq \frac{x_1 + \ldots + x_n}{n} \quad \text{and} \quad (x_{n+1} \ldots x_{2n})^{1/n} \leq \frac{x_{n+1} + \ldots + x_{2n}}{n}$$

with equality only if $x_1 = \ldots = x_n$ and $x_{n+1} = \ldots = x_{2n}$ respectively. Therefore, using monotonicity of the square root, we can continue $(*)$ to get

$$(x_1 x_2 \ldots x_{2n})^{1/(2n)} \leq \sqrt{\frac{x_1 + \ldots + x_n}{n} \times \frac{x_{n+1} + \ldots + x_{2n}}{n}}$$

with equality only if $x_1 = \ldots = x_n$ ($=: a$) and $x_{n+1} = \ldots = x_{2n}$ ($=: b$). Now using the base case, we continue

$$(x_1 x_2 \ldots x_{2n})^{1/(2n)} \leq \frac{1}{2}\left(\frac{x_1 + \ldots + x_n}{n} + \frac{x_{n+1} + \ldots + x_{2n}}{n}\right) = \frac{x_1 + \ldots x_{2n}}{2n}$$

with equality only if we had it in the previous step and now also have $\frac{x_1 + \ldots + x_n}{n} = \frac{x_{n+1} + \ldots + x_{2n}}{n}$ (i.e., $a = b$). In other words, now equality holds only when all x_i are equal.

(b) Now let n be arbitrary and choose $m > n$ to be a power of 2. We are given $x_1, \ldots, x_n > 0$ and define $x_{n+1} = \ldots = x_m := \frac{x_1 + \ldots + x_n}{n}$. Then by part (a), we have

$$\left(x_1 \cdots x_n \times \left(\frac{x_1 + \ldots + x_n}{n}\right)^{m-n}\right)^{1/m} \leq \frac{x_1 + \ldots + x_n + (m - n)\frac{x_1 + \ldots + x_n}{n}}{m}$$

with equality only if $x_1 = \ldots = x_n = \frac{x_1 + \ldots + x_n}{n}$. (The last of these equalities is redundant since it follows from the others.) Simplifying both sides, we get

$$(x_1 \cdots x_n)^{1/m} \times \left(\frac{x_1 + \ldots + x_n}{n}\right)^{(m-n)/m} \leq \frac{x_1 + \ldots + x_n}{n}$$

with equality only if $x_1 = \ldots = x_n$. Cancelling the power of the average from the left side, we get

$$(x_1 \cdots x_n)^{1/m} \leq \left(\frac{x_1 + \ldots + x_n}{n}\right)^{n/m} .$$

Taking the $\frac{m}{n}$th power proves the agm inequality for n. (The monotonicity of $x \mapsto x^q$ for $q > 0$ has followed immediately from its definition and the monotonicity of ln and exp.)

23: Young's inequality frequently comes in handy: "If $x, y > 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$." (This is often proved as a single variable calculus minimization problem, using the derivative.)

(a) Show that for $p > 1$, $L > 0$, the function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto \frac{1}{p}x^p + (1 - \frac{1}{p})(\frac{L}{x})^{p/(p-1)}$ has a minimum. (Hint: Start with some random x_0 (eg., $x_0 = 1$), then show there exists $[a, b] \subset \mathbb{R}_+$ such that for $x \in \mathbb{R}_+ \setminus [a, b]$, it holds $f(x) > f(x_0)$. Use the sequential compactness of $[a, b]$.)

(b) Assuming for the moment Calculus-1 knowledge about derivatives without having proved it, you can now find the location and value of the minimum.

(c) Alternatively, prove Young's inequality at least for rational p, q writing $p = \frac{m+n}{n}$ with $m, n \in \mathbb{N}$, as a consequence of the agm inequality. Then generalize to $p > 1$ real by using a continuity and limit argument.

Solution: (a) Choosing some arbitrary x_0 we let $f(x_0) =: y_0 > 0$.
Since $f(x) > (1 - \frac{1}{p})\,(L/x)^{p/(p-1)}$, there is some z_0 (namely $z_0 := [y_0/(1 - \frac{1}{p})]^{(p-1)/p}$) such that for $L/x > z_0$, we have $f(x) > y_0$. We choose $a := L/z_0 > 0$. So, if $x < a$, then $f(x) > y_0$. Similarly, since $f(x) > \frac{1}{p}x^p$, we can choose $b := (py_0)^{1/p}$ to get: If $x > b$, then $f(x) > y_0$.

Now f is continuous on $[a, b]$, so it takes a minimum there. Since $f(x_0) = y_0$, x_0 cannot be $< a$, nor $> b$, so $x_0 \in [a, b]$ and $m := \min\{f(x) : x \in [a, b]\} \leq f(x_0) = y_0$. Since for $x > b$ and $x < a$, we know $f(x) > y_0 \geq m$, too, f has a minimum over all of $\mathbb{R}_+$.

(b) Accepting the Calc 1 result that at a minimum, the derivative has to vanish, we can find the minimum among the solutions of $f'(x) = 0$. This is $x^{p-1} + (1 - \frac{1}{p})L^{p/(p-1)}(-\frac{p}{p-1})x^{-p/(p-1)-1} = 0$. This simplifies easily to $x = L^{1/p}$. And $m = f(L^{1/p}) = L$.

Young's inequality follows by letting $L := xy$. We also get from the uniqueness of the minimum that equality holds if and only if $x^p = y^q$.

(c) If $p > 1$ is rational, we can write it as $\frac{m+n}{n}$ for some $m, n \in \mathbb{N}$. The $q = (1 - \frac{1}{p})^{-1} = \frac{m+n}{m}$. Using the agm inequality with n copies of x^p and m copies of y^q, we get

$$\Big((x^p)^n\,(y^q)^m\Big)^{1/(m+n)} \leq \frac{n}{m+n}x^p + \frac{m}{m+n}y^q = \frac{1}{p}x^p + \frac{1}{q}x^q\ .$$

The left hand side simplifies to $(x^{m+n}y^{m+n})^{1/(m+n)} = xy$. Again, we get equality if and only if $x^p = y^q$, because this is when equality holds in agm.

Now let $p > 1$ be real. We can find a sequence p_j of rational numbers such that $p_j \to p$. (Consequence of the Archimedean property.) Then $q_j := (1 - \frac{1}{p_j})^{-1}$ converges to $(1 - \frac{1}{p})^{-1} = q$. Taking the limit on the right hand side of

$$xy \leq \frac{1}{p_j}x^{p_j} + \frac{1}{q_j}y^{q_j}$$

and using the continuity of the power function, we conclude $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$.

Note: The disadvantage of (c) over (b) is that due to the $\lim_{j\to\infty}$, we get $\leq$ in Young's inequality even for those x, y for which we had strict inequality for the p_j, q_j. So we lose the part "with equality only if $x^p = y^q$" when p, q are irrational.

24: Show that a sequentially compact metric space is complete. Also show by counterexample that the converse implication is false.

Solution: The converse is false, since $\mathbb{R}$ is complete but not sequentially compact: $(n)_{n\in\mathbb{N}}$ has no convergent subsequence since every subsequence is unbounded.

Now let (x_n) be a Cauchy sequence in a sequentially compact metric space X. It has a convergent subsequence (x_{n_j}), whose limit we call x_*.

Given $\varepsilon > 0$, we use the Cauchy property for $\frac{\varepsilon}{2}$ and get

$$\exists n_0 \, \forall n, m \geq n_0 : d(x_n, x_m) < \frac{\varepsilon}{2} \, .$$

We also use the limit property for $\frac{\varepsilon}{2}$ and get

$$\exists j_0 \, \forall j \geq j_0 : d(x_{n_j}, x_*) < \frac{\varepsilon}{2} \, .$$

It is no loss of generality to assume j_0 so large that $n_{j_0} \geq n_0$. Then the Cauchy property applies with $m := n_j$. We argue for $n \geq n_0$ that

$$d(x_n, x_*) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x_*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \, .$$

So $\lim x_n = x_*$.

25: Suppose (X, d) is a sequentially compact metric space and assume $f : X \to X$ is a weak contraction, i.e., $d(f(x), f(y)) < d(x, y)$ unless $x = y$. Prove that f has a unique fixed point. Hint for the existence part: Study the function $r : X \to \mathbb{R}\,,\ x \mapsto d(f(x), x)$ and show that it has a minimum. Assuming the minimum value is positive, obtain a contradiction.

Solution: Note that the contraction property implies continuity of f (with $\delta := \varepsilon$ in the definition. The function $r : x \mapsto d(f(x), x)$ is continuous, because f is continuous and d is, and compositions of continuous functions are continuous. As X is sequentially compact, r takes on a minimum there, say at x_0. Suppose $m := r(x_0) = \min r(x) : x \in X$ is positive. This means $d(f(x_0), x_0) > 0$, so $x_0 \neq f(x_0)$. But then the weak contraction property implies $d(f(f(x_0)), f(x_0)) < d(f(x_0), x_0) = m$. So we have a point $f(x_0) \in X$ that gives a smaller value than m to the function r. This contradicts the mnimality of m.

The contradiction shows that $m = 0$, so there exists some $x_0 \in X$ for which $r(x_0) = 0$, i.e., $f(x_0) = x_0$.

Variant: We can easily show the continuity of the function r directly: namely we have to show that for every $\varepsilon > 0$ there is a $\delta > 00$ such that $|r(x) - r(y)| < \varepsilon$, provided $d(x, y) < \delta$.

Now

$$r(x) - r(y) = d(f(x), x) - d(f(y), y) \leq \Big(d(f(x), f(y)) + d(f(y), y) + d(y, x)\Big) - d(f(y), y) \leq d(x, y) + d(y, x)$$

where we have used the weak contraction property in the last step. We can choose $\delta = \frac{1}{2}\varepsilon$ to conclude $r(x) - r(y) < \varepsilon$ if $d(x, y) < \delta$. By symmetry, the inequality $r(y) - r(x) < \varepsilon$ is shown analogously. (We have even proved uniform continuity.)

26: Show that every continuous function $f : [a, b] \to [a, b]$ has a fixed point (not necessarily unique).

Solution: Clearly $f(a) \geq a$ and $f(b) \leq b$. The function g given by $g(x) := f(x) - x$ is continuous and satisfies $g(a) \geq 0$ and $g(b) \leq 0$, so it has a zero in $[a, b]$. But zeros of g are fixed points of f.

Note: The following theorem (called Brouwer's Fixed Point Theorem) is a generalization that is useful to know. For now (and possibly ever), we are not proving it in this class. **Thm.:** Let B be a closed ball in $\mathbb{R}^n$ and $f : B \to B$ continuous. Then f has a fixed point.

(The theorem can be generalized to other Banach spaces, but needs another, major hypothesis to be added.)

27: Hölder inequality in $\mathbb{R}^n$: Prove, for $x_i, y_i \in \mathbb{R}$ with $i = 1, 2, \ldots, n$ and for $p, q > 1$ subject to $\frac{1}{p} + \frac{1}{q} = 1$, that $\sum_i |x_i y_i| \leq \left(\sum_i |x_i|^p\right)^{1/p} \left(\sum_i |y_i|^q\right)^{1/q}$. Hint: First prove it under the extra hypothesis $\sum_i |x_i|^p = 1 = \sum_i |y_i|^q = 1$, using Young's inequality; then conclude the general case by replacing x_i with λx_i and y_i with μy_i for appropriate λ, μ.

Solution: Using Young's inequality on $|x_i|$ and $|y_i|$, we have $|x_i y_i| \leq \frac{1}{p}|x_i|^p + \frac{1}{q}|y_i|^q$. Summing over i gives

$$\sum_i |x_i y_i| \leq \frac{1}{p}\Big(\sum_i |x_i|^p\Big) + \frac{1}{q}\Big(\sum_i |y_i|^q\Big) .$$

If $\sum |x_i|^p = 1$ and $\sum |y_i|^q = 1$, then both Young (proved) and Hölder (to be shown) claim the same, namely $\sum |x_i y_i| \leq 1$. So Hölder is proved under this hypothesis.

Given arbitrary x_i, not all 0, we let $\lambda := (\sum |x_i|^p)^{1/p}$; given arbitrary y_i, not all 0, we let $\mu := (\sum |y_i|^q)^{1/q}$; then x_i/λ and y_i/μ satisfy the extra hypothesis under which Hölder was just proved. So we have

$$\sum_i \frac{|x_i y_i|}{\lambda\mu} \leq \Big(\sum_i |x_i/\lambda|^p\Big)^{1/p} \Big(\sum_i |y_i/\mu|^q\Big)^{1/q}$$

Multiplying by $\lambda\mu$, proves Hölder in the general case, except when all x_i are 0 or all y_i are 0. However, if all x_i are zero, or all y_i are zero, Hölder reduces to $0 \leq 0$ and is obviously true.

28: Let $p > 1$. For $x \in \mathbb{R}^n$, define $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$. Show that $\|\cdot\|_p$ is a norm. Hint: to prove the triangle inequality, observe that $|x_i + y_i|^p \leq |x_i + y_i|^{p-1}|x_i| + |x_i + y_i|^{p-1}|y_i|$ and use Hölder's inequality. Note: The triangle inequality for this norm is also known under the name Minkowski's inequality; the correct pronounciation of the name is minn-COUGH-skee, even though anglicized pronounciations are frequently encountered.

Solution: The properties $\|x\|_p \geq 0$, and $\|x\|_p = 0 \Longrightarrow x = 0$, and $\|\lambda x\|_p = |\lambda|\|x\|_p$ are trivial. For the triangle inequality, we multiply $|x_i + y_i| \leq |x_i| + |y_i|$ with $|x_i + y_i|^{p-1}$, and then sum over i, we get

$$\sum_i |x_i + y_i|^p \leq \sum_i |x_i + y_i|^{p-1}|x_i| + \sum_i |x_i + y_i|^{p-1}|y_i| .$$

Using Hölder's inequality on each summand on the right, with exponent p on the 2nd factor and exponent $q = \frac{p}{p-1}$ on the first factor, we conclude

$$\sum_i |x_i + y_i|^p \le \left(\sum_i |x_i + y_i|^p\right)^{(p-1)/p} \left(\sum_i |x_i|^p\right)^{1/p} + \left(\sum_i |x_i + y_i|^p\right)^{(p-1)/p} \left(\sum_i |y_i|^p\right)^{1/p} .$$

Cancelling the common factor proves

$$\left(\sum_i |x_i + y_i|^p\right)^{1/p} \le \left(\sum_i |x_i|^p\right)^{1/p} + \left(\sum_i |y_i|^p\right)^{1/p} \text{, i.e.,} \quad \|x + y\|_p \le \|x\|_p + \|y\|_p .$$

29: (a) Show that $\mathcal{T} := \{]-\infty, a[\, : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is a topology on $\mathbb{R}$. Use the separation property (namely that for any two distinct points x, y in a metric space one can find disjoint r-balls $U(x, r)$ and $U(y, r)$) to show that this topology is not generated by a metric (i.e., is not the family of open sets defined in terms of any metric on $\mathbb{R}$).
(b) analogously, $\mathcal{T}' := \{]a, \infty[\, : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is a topology. No need to redo part (a) for this one; just mentioning it.

Solution: (a) It is trivial that $\emptyset, \mathbb{R} \in \mathcal{T}$. As for finite intersections, let $U_i \in \mathcal{T}$ for $i = 1, \ldots, n$. If any of the U_i is the empty set, then so is the intersection and we are done. Otherwise, if all of the U_i are $\mathbb{R}$, the intersection is $\mathbb{R}$ and we are done again. So we may assume that some of the U_i are of the form $]-\infty, a_i[$ with $a_i \in \mathbb{R}$, and none are empty. If there are also some $U_i = \mathbb{R}$ in the list, they do not affect the intersection; so it is no loss of generality to assume $U_i =]-\infty, a_i[$ for all a_i. Then $\bigcap_{i=1}^n U_i =]-\infty, a[$ where $a = \min_{i=1}^n a_i$. Note: It suffices to show that $U, V \in \mathcal{T} \Longrightarrow U \cap V \in \mathcal{T}$, since the case of n sets follows by induction.

Now let U_λ for $\lambda \in \Lambda$ be an arbitrary family of sets from $\mathcal{T}$. We need to show $U := \bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$. If any of the U_λ is $\mathbb{R}$, the union is $\mathbb{R}$ and we are done. If all U_λ are empty, the union is empty and we are done again. So we may assume that none are $\mathbb{R}$ and some are of the form $U_\lambda =]-\infty, a_\lambda[$; we may ignore any remining U_λ that are empty, because they do not affect the union. So it is no loss of generality to assume $U_\lambda =]-\infty, a_\lambda[$ for all $\lambda \in \Lambda$.

If $S := \{a_\lambda : \lambda \in \Lambda\}$ is unbounded, the union is $\mathbb{R}$ because any $x \in \mathbb{R}$ is *not* a bound for S, so we find $\lambda \in \Lambda$ with $a_\lambda > x$; and then $x \in U_\lambda \subset U$. On the other hand, if S is bounded, let $a := \sup S \in \mathbb{R}$. We claim $U =]-\infty, a[$. Indeed, since $a \ge a_\lambda$ for all λ, we have $U_\lambda \subset]-\infty, a[$, and therefore $U \subset]-\infty, a[$. Conversely, if $x \in]-\infty, a[$, then $x < a$, and x is *not* an upper bound for S (because a was the smallest upper bound). So there exists an $a_\lambda > x$; but then $x \in U_\lambda \subset U$.

We have proved that $\mathcal{T}$ is a topology on $\mathbb{R}$.

Clearly, if $U, V \in \mathcal{T}$ and $U \ne \emptyset, V \ne \emptyset$, then $U \cap V \ne \emptyset$. On the other hand, if $\mathcal{T}$ consisted of the open sets defined by a metric, then (using that $\mathbb{R}$ has at least two distinct points x, y), we could find disjoint nonempty open sets $U, V \in \mathcal{T}$; namely open balls of radius $< \frac{1}{2} d(x, y)$ centered at x and y respectively. So $\mathcal{T}$ is not generated by a metric.

30: With the topology from the previous problem chosen on $\mathbb{R}$, prove that

$$f : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, \mathcal{T})$$

has the property "V open $\Longrightarrow f^{-1}(V)$ open" if and only if $\limsup_{x\to x_0} f(x) \le f(x_0)$ for all x_0. — Def: We call such functions upper semicontinuous.

Solution: Assume f has the property "V open $\Longrightarrow f^{-1}(V)$ open". For given x_0, we define $s := \limsup_{x\to x_0} f(x)$ and want to show that $f(x_0) \ge s$. Writing out the definition of lim sup, we have

$$s := \lim_{\varepsilon\to 0} s_\varepsilon \quad \text{where} \quad s_\varepsilon := \sup\{f(x) : |x - x_0| < \varepsilon\} .$$

Since $\varepsilon \mapsto s_\varepsilon$ is nonincreasing, we know $s_\varepsilon \ge s$ for all ε.

Now assume, for the sake of contradiction, that $f(x_0)$ were $< s$, so we find an s' between $f(x_0)$ and s, namely we have: $f(x_0) < s' < s$. So $f(x_0) \in \,]-\infty, s'[\, =: V \in \mathcal{T}$; then we know from the assumed property of f that $f^{-1}(V)$ is open, i.e., with $x_0 \in f^{-1}(V)$, an entire open ball $U(x_0, \varepsilon)$ is contained in $f^{-1}(V)$. But then all x in this ε-ball satisfy $f(x) < s'$, hence their sup of these $f(x)$, namely s_ε, is still $\le s'$, and thus $s \le s_\varepsilon \le s' < s$, a contradiction.

This contradiction has proved that $\limsup_{x\to x_0} f(x) \le f(x_0)$.

Now conversely, we assume that f satisfies the property $\limsup_{x\to x_0} f(x) \le f(x_0)$ for all x_0. We need to conclude the property "$f^{-1}(V)$ open for $V \in \mathcal{T}$". We may assume $V = \,]-\infty, a[$ with $a \in \mathbb{R}$, since the property is trivially true when $V = \emptyset$ or $V = \mathbb{R}$.

So let $x_0 \in f^{-1}(]-\infty, a[)$, i.e., $f(x_0) < a$. Then $\limsup_{x\to x_0} f(x) \le f(x_0) < a$. But by the definition of lim sup, this implies that there is an $\varepsilon > 0$ for which $s_\varepsilon < a$, where s_ε is as defined above. But then all $x \in U(x_0, \varepsilon)$ satisfy $f(x) \le s_\varepsilon < a$ and are therefore in $f^{-1}(]-\infty, a[)$. So we have showed that $f^{-1}(V)$ is open when $V = \,]-\infty, a[$. As mentioned before, the cases $V = \emptyset$ and $V = \mathbb{R}$ are trivial.

31: In a metric space (X, d), let A be a closed set and define $f(x) := \text{dist}(A, x) := \inf\{d(x, z) : z \in A\}$. Show that $f : X \to \mathbb{R}$ is continuous and that $f^{-1}(\{0\}) = A$. (In particular, any closed set can occur as the inverse image of a singleton under a continuous function.)

Solution: First, to show the continuity we show that $|\text{dist}(A, x) - \text{dist}(A, y)| \le d(x, y)$: Indeed for any $z \in A$, it holds $d(x, z) \le d(x, y) + d(y, z)$. Tanking the inf over $z \in A$ on both sides, we obtain $\text{dist}(A, x) \le d(x, y) + \text{dist}(A, y)$, hence $\text{dist}(A, x) - \text{dist}(A, y) \le d(x, y)$. The same argument applies with x and y swapped. The closedness of A was not needed in this argument.

Next we have to show $\text{dist}(A, x) = 0$ iff $x \in A$. The 'if' part is trivial because we can choose $z = x$ in the definition of dist. For the converse, we assume $\text{dist}(A, x) = 0$ and want to show $x \in A$. So $\inf\{d(x, z) : z \in A\} = 0$, and therefore for every $n \in \mathbb{N}$, there is some $z_n \in A$ for which $d(z_n, x) < \frac{1}{n}$. This implies that the sequence (z_n) has limit x. Since $(z_n) \subset A$ and A is closed, its limit x is also in A.

32: Give an example of a continuous $f : \mathbb{R} \to \mathbb{R}$ (with the usual topology defined by the metric), and a connected $C \in \mathbb{R}$, such that $f^{-1}(C)$ is not connected.

Solution: We can for instance take $f(x) = x^2$ and $C = [1, 4]$. Then $f^{-1}(C) = [-2, -1] \cup [1, 2]$ is not connected.

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