**Homework**
**UTK – M447 – Honors Advanced Calculus I – Fall 2015**
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**1.** Prove that for every $a \in \mathbb{R}_+$, there exists a unique $x \in \mathbb{R}_+$ for which $x^2 = a$. Specifically:
(a) Show uniqueness of $x$; also obtain the lemma "If $x, y > 0$ and $x^2 = a$ and $y^2 = b$, and $a < b$, then $x < y$" in this proof.
(b) Letting $S := \{y \in \mathbb{R} \mid y > 0 \text{ and } y^2 \le a\}$, show that $S$ is non-empty and bounded above.
(c) Defining $x := \sup S$, show that for every $\varepsilon > 0$, $a - \varepsilon < x^2 < a + \varepsilon$, then conclude $x^2 = a$.

**2.** For $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, define $|z| := \sqrt{x^2 + y^2}$. Prove that $|z + w| \le |z| + |w|$. Make sure that any two inequalities you write down are properly connected with a logical direction of implication (verbally or in symbols: $\Longrightarrow$ or $\Longleftarrow$ or $\Longleftrightarrow$ ), and that the logical direction of implication matches the needs for the proof.

*You may only use ordered field axioms, simple conclusions from them (like subtraction on both sides of inequalities, or '$a < b$ and $c < d$ implies $a + c < b + d$' or '$0 < a < b$ and $0 < c < d$ implies $0 < ab < cd$' and their analogs with $\le$), the supremum axiom and simple conclusions thereof, and results from the previous problem about the square root.*

**3.** Prove: If $\lim a_n = a_*$ and $\lim b_n = b_*$ and $b_* \neq 0$, then $\lim(a_n/b_n) = a_*/b_*$.

**4.** Prove (directly from the axioms and consequences proved in class) that $\lim \frac{1}{n} = 0$. (That should be just a few lines.)

**5.** Assuming $(a_n)$ and $(b_n)$ are bounded sequences of real numbers: Is the following statement true or false? "¿$\limsup(a_n + b_n) = \limsup a_n + \limsup b_n$?" If true, provide a proof; if false, provide a counterexample, and if possible, prove an amendment replacing '=' with either '$\le$' or '$\ge$' that results in a true statement. If no such amendment can be proved, provide counterexamples against the amended versions.

**6.** Prove by induction the lemma: If $y > -1$ and $n \in \mathbb{N}$, then $(1 + y)^n \ge 1 + ny$. (You'll need it later.)

**7.** *Note:* The power laws $(ab)^n = a^n b^n$, $(a/b)^n = a^n/b^n$ and $a^{n+m} = a^n a^m$ are easy consequences from the field axioms, and induction. You may use them without providing their proof.
(a) Given $x \in \mathbb{R}$, consider the sequence $(a_n)$ given by $a_n := (1 + \frac{x}{n})^n$. We will later write $a_n(x)$ for $a_n$, when the dependence on $x$ plays a role. Show that $a_{n+1}/a_n \ge 1$ whenever $x \ge 0$ or $n > |x| + 1$. (The previous lemma may help in estimating $[(1 + \frac{x}{n+1})/(1 + \frac{x}{n})]^n$. )
(b) Show that the sequence $(a_n)$ is bounded above.
*Hint:* There may be a variety of ways; but you could use that $a_n(x)a_n(-x) < 1$ for $|x| < 1$ and a lower bound of $a_n(x)$ for $-1 < x < 0$ to get an upper bound for $0 < x < 1$. And maybe $a_n(x + y) < a_n(x)a_n(y)$ for $x > 0$.
(c) Defining $\exp(x) := \lim_{n\to\infty} a_n(x)$, prove that, for all $x, y \in \mathbb{R}$, it holds $\exp(x)\exp(-x) = 1$ and $\exp(x + y) = \exp(x)\exp(y)$. Also prove $x < y \Longrightarrow \exp(x) < \exp(y)$.

**8.** Let $(a_n)$ be a sequence in $\mathbb{R}$ that is bounded below and 'sub-additive', i.e., $\forall n, m : a_{n+m} \le a_n + a_m$. Prove that $\lim \frac{a_n}{n}$ exists. *Hints follow:*
(a) Prove for $k, n \in \mathbb{N}$ that $a_{kn} \le k a_n$.

(b) Prove for $n, r \in \mathbb{N}$ that $\limsup_{k\to\infty} \frac{a_{kn+r}}{kn+r} \le \frac{a_n}{n}$.
(c) You may assume the 'division with remainder theorem from elementary number theory': $\forall m, n \in \mathbb{N}\ \exists k, r \in \mathbb{N}_0 : m = kn + r$ and $0 \le r \le n-1$. Use it to prove $\limsup \frac{a_m}{m} \le \inf \frac{a_n}{n}$ and conclude the original claim.

**9.** Using the dot product in $\mathbb{R}^n$, namely $\vec{u}\cdot\vec{v} := \sum_{i=1}^n u_i v_i$, we can write $\|\vec{u}\|_2^2 = \vec{u}\cdot\vec{u}$.
(a) Prove the Cauchy-Schwarz inequality $|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|_2\,\|\vec{v}\|_2$ by exploiting the fact that $(\vec{u}+t\vec{v})\cdot(\vec{u}+t\vec{v}) \ge 0$ for all $t \in \mathbb{R}$ and then choosing a special $t$. (Actually the one that, according to Calculus 1, minimizes the given expression. But note that, logically, you do not need to justify your choice of $t$.)
(b) Use it to conclude $\|\vec{u}+\vec{v}\|_2 \le \|\vec{u}\|_2 + \|\vec{v}\|_2$.
*[too elementary: I won't collect this one for grading.]*

**10.** Prove the triangle inequality for $\|\cdot\|_\infty$ on $\mathcal{BF}(X \to \mathbb{R}) = \{f : X \to \mathbb{R} : f \text{ bounded}\}$. Here $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$ and $X$ is any set.

**11.** I mentioned the norms $\|u\|_p := \left(\sum_{i=1}^n |u_i|^p\right)^{1/p}$ without proof of the norm properties, where $p \ge 1$. Now try $p = \frac{1}{2}$. Is it a norm? Proof or counterexample.

**12.** Assume $d$ is a distance function on $X$. Show that $d_1 := \frac{d}{1+d}$ is also a distance function. Show that a sequence is convergent with respect to $d$ if and only if it is convergent with respect to $d_1$.

**13.** On $\mathbb{N}$ a distance function $d_1$ is given by $d_1(n,m) := |n-m|$. Another distance function is given by $d_2(n,m) := |\frac{1}{n} - \frac{1}{m}|$. Which are the convergent sequences in each case? Which are Cauchy sequences in each case?

**14.** The following example is instructive, albeit not significant for purposes of Calculus: You may and should use basic facts about prime factor decompositions of rational numbers in this example. Given a rational $q = \frac{m}{n} \in \mathbb{Q}$ with $m \in \mathbb{Z}, n \in \mathbb{N}$, we consider the prime factor decomposition $q = \pm p_1^{k_1} \cdots p_r^{k_r}$. Choosing a particular prime $p$, we define $[0]_p := 0$; and for $q \ne 0$, we define $[q]_p := 1/p^k$, where $p^k$ is the power of $p$ that occurs in the prime factor decomposition of $q$. *Example:* $[\frac{12}{25}]_3 = \frac{1}{3}$, $[\frac{2}{9}]_3 = 9$, $[\frac{2}{5}]_3 = [1]_3 = 1$.

Prove the strengthened triangle inequality $[x+y]_p \le \max\{[x]_p, [y]_p\}$ and show that $d_p(x,y) := [x-y]_p$ defines a metric on $\mathbb{Q}$. (It is called the $p$-adic metric).

Let $a_n := \frac{2^n}{2^n+1}$. Find the limit $\lim a_n$ in $\mathbb{Q}$ with the metric $d(x,y) := |x-y|$ (that is too easy). Next find $\lim a_n$ with respect to the metric $d_2$.

**15.** Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that does not have a fixed point, but satisfies a weakened contraction condition $|f(x) - f(y)| < |x-y|$ whenever $x \ne y$. For this example, you may use Calculus 1 knowledge (even though I think it can also be done with the material constructed in this class already.)

**16.** Prove that $f : [0,\infty[\ \to \mathbb{R},\ x \mapsto \sqrt{x}$ is uniformly continuous. (Of course continuity is understood here with respect to the usual distance on $\mathbb{R}$.)

**17.** Let $(X,d)$ be the metric space of $\mathbb{R}$ with the *discrete* metric $d(x,y) := 1$ whenever $x \ne y$. Let $(Y,d)$ be *any* metric space, and $\mathbb{R}$ be equipped with the usual metric.

(a) Show that every function $f : X \to Y$ is continuous. Is the same true with 'uniformly continuous'?
(b) Conjecture which functions $f : \mathbb{R} \to X$ are continuous (we'll prove it in class) .

**18.** Suppose $(X, d_X)$ and $(Y, d_Y)$ are metric spaces: the set $X \times Y$ can be made into a metric space by defining $d_\infty((x, y), (x', y')) := \max\{d_X(x, x'), d_Y(y, y')\}$.
(a) Prove this.
(b) With this definition of the metric on $X \times X$, write out what the statement "$d_X : X \times X \to \mathbb{R}$ is continuous" means and then prove it. Decide if the same statement is true with 'uniformly continuous' instead of merely 'continuous'.

**19.** Let $g : X \to Y$ and $f : Y \to Z$ be functions, where $X$, $Y$, $Z$ are metric spaces; let $f \circ g : X \to Z\,,\ x \mapsto f(g(x))$ be their composition. If $g$ is continuous at $x_0$ and $f$ is continuous at $y_0 := g(x_0)$, show that $f \circ g$ is continuous at $x_0$.

**20.** Assume $X$ and $Y$ are metric spaces, where the metric on $Y$ is bounded. Per #12, boundedness of the metric is no loss of generality. We can define $\mathcal{F}(X \to Y)$ to be the set of all functions from $X$ to $Y$ and make it into a metric space by defining $d_\infty(f, g) := \sup\{d(f(x), g(x)) : x \in X\}$. (You don't need to prove these simple facts; they are similar to #10.)

Show that convergence $f_n \to f$ in the sense of the metric $d_\infty$ is equivalent to uniform convergence $f_n \to f$.

We use the metric $d_\infty$ on $C^0(X \to Y)$, the subset of *continuous* functions from $X$ to $Y$. Show this metric space is complete, if $Y$ is complete.

**21.** (a) Prove that $\exp : \mathbb{R} \to \mathbb{R}_+$ as defined previously is bijective.
(b) We'll denote its inverse function as ln, so $y = \ln x :\iff x = \exp(y)$. Prove that $\ln : \mathbb{R}_+ \to \mathbb{R}$ is increasing and continuous. Also prove $\ln(xy) = \ln x + \ln y$.
(c) For $x > 0$ and $q \in \mathbb{R}$, we define $x^q := \exp(q \ln x)$. Prove for all $x > 0$ and $q_1, q_2 \in \mathbb{R}$ that $x^{q_1+q_2} = x^{q_1} x^{q_2}$ and $x^{q_1 q_2} = (x^{q_1})^{q_2}$. Also prove $(xy)^q = x^q y^q$ for all $x, y > 0$ and $q \in \mathbb{R}$. Note that $(x, q) \mapsto x^q$, $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous.
(d) letting $e := \exp(1)$, show that $\exp(x) = e^x$

**22.** Having constructed arbitrary powers, it's time to introduce the inequality of the arithmetic and geometric mean (short: agm inequality): "If $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}_+$, then $(x_1 x_2 \cdots x_n)^{1/n} \le \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$. The inequality is strict unless $x_1 = \ldots = x_n$."

(a) Prove the inequality for $n = 2^k$ by induction over $k$.
(b) Then prove it for arbitrary $n$ by choosing some $k$ such that $m := 2^k > n$ and defining $x_{n+1}, \ldots, x_m$ wisely.

(I won't collect this for grading, but if you're not familiar with it, you should do it, as it is a 'must know'.)

**23.** Young's inequality frequently comes in handy: "If $x, y > 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $xy \le \frac{1}{p}x^p + \frac{1}{q}y^q$." (This is often proved as a single variable calculus minimization problem, using the derivative.)

(a) Show that for $p > 1$, $L > 0$, the function $f : \mathbb{R}_+ \to \mathbb{R}_+\,,\ x \mapsto \frac{1}{p}x^p + (1 - \frac{1}{p})(\frac{L}{x})^{p/(p-1)}$ has a minimum. (Hint: Start with some random $x_0$ (eg., $x_0 = 1$), then show there exists $[a, b] \subset \mathbb{R}_+$ such that for $x \in \mathbb{R}_+ \setminus [a, b]$, it holds $f(x) > f(x_0)$. Use the sequential compactness of $[a, b]$. )

(b) Assuming for the moment Calculus-1 knowledge about derivatives without having proved it, you can now find the location and value of the minimum.

(c) Alternatively, prove Young's inequality at least for rational $p, q$ writing $p = \frac{m+n}{n}$ with $m, n \in \mathbb{N}$, as a consequence of the agm inequality. Then generalize to $p > 1$ real by using a continuity and limit argument.

**24.** Show that a sequentially compact metric space is complete. Also show by counterexample that the converse implication is false.

**25.** Suppose $(X, d)$ is a sequentially compact metric space and assume $f : X \to X$ is a weak contraction, i.e., $d(f(x), f(y)) < d(x, y)$ unless $x = y$. Prove that $f$ has a unique fixed point. Hint for the existence part: Study the function $r : X \to \mathbb{R},\ x \mapsto d(f(x), x)$ and show that it has a minimum. Assuming the minimum value is positive, obtain a contradiction.

**26.** Show that every continuous function $f : [a, b] \to [a, b]$ has a fixed point (not necessarily unique).

**27.** Hölder inequality in $\mathbb{R}^n$: Prove, for $x_i, y_i \in \mathbb{R}$ with $i = 1, 2, \ldots, n$ and for $p, q > 1$ subject to $\frac{1}{p} + \frac{1}{q} = 1$, that $\sum_i |x_i y_i| \le \left(\sum_i |x_i|^p\right)^{1/p} \left(\sum_i |y_i|^q\right)^{1/q}$. Hint: First prove it under the extra hypothesis $\sum_i |x_i|^p = 1 = \sum_i |y_i|^q = 1$, using Young's inequality; then conclude the general case by replacing $x_i$ with $\lambda x_i$ and $y_i$ with $\mu y_i$ for appropriate $\lambda, \mu$.

**28.** Let $p > 1$. For $x \in \mathbb{R}^n$, define $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$. Show that $\|\cdot\|_p$ is a norm. Hint: to prove the triangle inequality, observe that $|x_i + y_i|^p \le |x_i + y_i|^{p-1}|x_i| + |x_i + y_i|^{p-1}|y_i|$ and use Hölder's inequality. Note: The triangle inequality for this norm is also known under the name Minkowski's inequality; the correct pronounciation of the name is minn-COUGH-skee, even though anglicized pronounciations are frequently encountered.

**29.** (a) Show that $\mathcal{T} := \{]-\infty, a[\ : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is a topology on $\mathbb{R}$. Use the separation property (namely that for any two distinct points $x, y$ in a metric space one can find disjoint $r$-balls $U(x, r)$ and $U(y, r)$) to show that this topology is not generated by a metric (i.e., is not the family of open sets defined in terms of any metric on $\mathbb{R}$).
(b) analogously, $\mathcal{T}' := \{]a, \infty[\ : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is a topology. No need to redo part (a) for this one; just mentioning it.

**30.** With the topology from the previous problem chosen on $\mathbb{R}$, prove that

$$f : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, \mathcal{T})$$

has the property "$V$ open $\Longrightarrow f^{-1}(V)$ open" if and only if $\limsup_{x \to x_0} f(x) \le f(x_0)$ for all $x_0$. — Def: We call such functions upper semicontinuous.

**Clarification:** The notation $f : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, \mathcal{T})$ means that the domain of $f$ is the metric space $\mathbb{R}$, equipped with the usual metric defined by the absolute value. In contrast, the target set is the set of real numbers with the topology $\mathcal{T}$ (i.e., it is not seen as a metric space). By definition, a set $V \in (\mathbb{R}, \mathcal{T})$ will be called open, iff $V \in \mathcal{T}$. As set $U \in (\mathbb{R}, |\cdot|)$ is called open according to the definition of open in metric spaces.

**31.** In a metric space $(X, d)$, let $A$ be a closed set and define $f(x) := \text{dist}(A, x) := \inf\{d(x, z) : z \in A\}$. Show that $f : X \to \mathbb{R}$ is continuous and that $f^{-1}(\{0\}) = A$. (In particular, any closed set can occur as the inverse image of a singleton under a continuous function.)

**32.** Give an example of a continuous $f : \mathbb{R} \to \mathbb{R}$ (with the usual topology defined by the metric), and a connected $C \subset \mathbb{R}$, such that $f^{-1}(C)$ is not connected.

**33.** (a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and a compact set $K$ such that $f^{-1}(K)$ is not compact.
(b) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ which maps compact sets into compact sets, but which is not continuous.

**34.** Show that every compact metric space $K$ contains a dense sequence, i.e., a sequence $(a_n)$ such that the closure of the set $\{a_n : n \in \mathbb{N}\}$ is $K$.

**35.** In a metric space $X$, let $C_1 \supset C_2 \supset C_3 \supset \ldots$ be a descending sequence of connected and non-empty sets (the notion 'descending' is defined by the given set inclusions). Let $C_* := \bigcap_{n=1}^{\infty} C_n$.
(a) Show by counterexamples, eg., in $\mathbb{R}^2$, that $C_*$ need not be non-empty, and also that $C_*$ need not be connected.
(b) Show: if $(C_n)$ is a descending sequence of compact connected non-empty sets, then $C_* := \bigcap_{n=1}^{\infty} C_n$ is also compact, connected, and non-empty. *Hint: for the connectedness, assume $C_* = \Omega_*^1 \cup \Omega_*^2$, a disjoint union of non-empty open sets. Define $\Omega_n^1 := (X \setminus \Omega_*^2) \cap C_n$ and $\Omega_n^2 := (X \setminus \Omega_*^1) \cap C_n$.*

**36.** Show: If a function $f : \mathbb{R}^n \to \mathbb{R}^m$ has both the properties "$K$ compact $\Longrightarrow f(K)$ compact" and "$C$ connected $\Longrightarrow f(C)$ connected", then $f$ is necessarily continuous.
*Hint: To show continuity at 0 (with no loss of generality), let $B_n := B(0, \frac{1}{n})$ and consider $f(B_n) =: D_n$. Use the previous hwk to learn about $\bigcap D_n =: D_*$. Explain why $D_*$ must consist of either one or infinitely many points. Then, assuming $D_*$ is infinite, construct a sequence $(y_n)$ of distinct points in $D_* \setminus \{f(0)\}$ that converges to a point $y_*$ that is neither $f(0)$ nor any of the $y_n$. From $x_n$ satisfying $y_n = f(x_n)$, construct a compact set whose image under $S$ is not compact. Conclude continuity from $D_*$ being a singleton.*

**37.** Show that $f : X \to \mathbb{R}$ is continuous exactly if it is both lower and upper semi-continuous.

**38.** Assume $f : K \to \mathbb{R}$ is lower semicontinuous and $K$ is compact. Show in two ways that $f$ takes on a minimum; namely using sequential compactness, then using cover-compactness. — Also give an example of a lower semi-continuous function $f : K \to \mathbb{R}$ that does *not* take on a maximum.

**39.** Assume $f : X \to Y$ is continuous and injective (not necessarily surjective); here $X, Y$ are metric spaces. We wonder whether the inverse function $f^{-1} : f(X) \to X$ is continuous as well:
(a) Show: If $X$ is compact, then we can conclude that $f^{-1}$ is continuous.
(b) Show: If $X$ is an interval in $\mathbb{R}$, and $Y = \mathbb{R}$, then we can conclude that $f^{-1}$ is continuous.
(c) Give a counterexample where $X = [0, 2\pi[$ and $Y = \mathbb{R}^2$ where $f^{-1}$ is not continuous.
(d) Give a counterexample with $X \subset \mathbb{R}$ and $Y = \mathbb{R}$, where $f^{-1}$ is not continuous.

**40.** Show that for a set $M$ in a metric space $X$, its interior $\mathring{M}$, which we had defined as $M \setminus \partial M$ where $\partial M = \{x \in X : \forall \varepsilon > 0 : U(x, \varepsilon) \cap M \neq \emptyset \wedge U(x, \varepsilon) \cap (X \setminus M) \neq \emptyset\}$, is the union of all open subsets of $M$, and thus is the largest open subset of $M$. Also, show that the closure $\bar{M}$, which we had defined as $M \cup \partial M$, is the intersection of all closed sets that contain $M$.

**41.** Find a set $M \subset \mathbb{R}$ for which all of the following sets are different:
$M$, $\mathring{M}$, $\bar{M}$, $\mathring{\bar{M}}$, $\overline{\mathring{M}}$, $\overline{\mathring{\bar{M}}}$, $\mathring{\overline{\mathring{M}}}$.