

Here are homework for 3.4 that I'd like to collect for grading:

Exercise **3.34** from notes.

3.4.J1: Study Prop. 3.4.11 in notes and prove the following generalization: Given $A \subset X$, $A \neq \emptyset$ in a metric space (X, d) , define $\text{dist}(x, A) := \inf\{d(x, y) \mid y \in A\}$. Prove that $|\text{dist}(x_1, A) - \text{dist}(x_2, A)| \leq d(x_1, x_2)$, and conclude that for given A , the function $\text{dist}(\cdot, A) : X \rightarrow \mathbb{R}$ that assigns to each x the value $\text{dist}(x, A)$ is continuous

3.4.J2: (In this problem you may use properties of the integral from elementary calculus.) Let $X := C^0[a, b]$, the set of all continuous functions $[a, b] \rightarrow \mathbb{R}$, with the sup distance as in Example 3.1.4. Show that the function $I : C^0[a, b] \rightarrow \mathbb{R}$ given by $I(f) := \int_a^b f(x) dx$ is continuous.

3.4.J3: (In this problem you may use knowledge about the derivative from elementary calculus.) Let $A := C^1[a, b]$, the set of all once continuously differentiable functions from $[a, b]$ to \mathbb{R} . Consider A as a subset of the metric space $X := C^0[a, b]$ and study the function $D : A \rightarrow X$ defined by $D(f) = f'$. Is D continuous? (Proof or counterexample.)

The followings pertain to 'Examples of metric spaces' (and has been postponed until here because I only now introduced dist .) Let X be a metric space, A a subset of X , and $x \in X$

3.4.J4: Suppose A is closed. Then $\text{dist}(x, A) = 0$ iff $x \in A$.

3.4.J5: If X is a metric space 'with finite diameter', i.e., there exists M such that $d(x, y) < M$ for all $x, y \in X$. Let $\text{Cl}(X)$ be the set of all closed subsets of X . On $\text{Cl}(X)$, define

$$d_H(A, B) := \max\{\sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A)\}$$

Show that d_H is a metric on $\text{Cl}(X)$ (called the Hausdorff distance). Hint for proving the triangle inequality. Let $a \in A$, $b \in B$, $c \in C$ and start with $d(a, b) + d(b, c) \geq d(a, c)$. Now carefully and judiciously, one step and one term at a time, in wisely chosen order, apply the definitions of dist and the upper / lower bound parts of the definitions of \sup and \inf until you get the triangle inequality for d_H .

(**Note:** This example is *sometimes* useful, because it provides a framework within which one could state that a quantity that depends on a *set* depends continuously on it. However, the usefulness of d_H is also limited, because many useful quantities (like for instance volume of a closed and bounded set in \mathbb{R}^3) do *not* depend continuously on the set if d_H is used to define continuity. For instance, the sequence of finite sets $A_n := \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ converges to $[0, 1]$ in the sense of the Hausdorff distance.