

## Addenda to Course Notes for 447

### 2.5a Other fields

We are mainly interested in the fields of real and complex numbers (and the field of rational numbers). However, the following two fields serve the purpose of better fathoming the scope of the field axioms.

**Example 2.5.x1:** *There is a field with exactly two elements,  $\mathbb{F} = \{e, o\}$ , subject to the rules  $e + e = e$ ,  $e + o = o + e = o$ ,  $o + o = e$ ,  $ee = e$ ,  $eo = oe = e$ , and  $oo = o$ . Checking the field axioms is routine, albeit many cases need to be considered.  $e$  plays the role of 0, and  $o$  plays the role of 1. In particular, this example shows that you cannot prove  $1 + 1 \neq 0$  from the field axioms alone. We can use the order axioms on top of the field axioms to prove that  $1 + 1 \neq 0$  in any ordered field. — The motivation behind this example is that  $o$  stands for ‘odd integer’ and  $e$  stands for ‘even integer’, and the definitions in this field incorporate properties of arithmetic of integers. However, this is just the motivational part, and of course you may not translate the property  $o^{-1} = o$  in the field into a claim that the reciprocal of an odd integer is an odd integer (an obviously false statement). The meaning of  $^{-1}$  comes from the precise wording of the field axioms, not from the motivation for this field.*

The following example gives an ordered field that does not satisfy the supremum axiom (that alone would be easy, take  $\mathbb{Q}$ ), but also does not satisfy the Archimedean property, which we had derived as a consequence of the supremum axiom. [Incidentally, the archimedean field  $\mathbb{Q}$  shows that the supremum axiom is stronger than the archimedean property, i.e., cannot be proved by assuming the archimedean property as an axiom instead.]

The field in question is one whose elements are familiar to you from elementary calculus; however to prove that it satisfies the properties, we need to rely on results from calculus, which we have not rigorously re-proved yet within our axiomatic setting. Since we are not relying on this example in our logical foundation of calculus, this is no problem. You could quarantine the example away unused for now and restudy it later. But I present it here, because I believe that seeing such an example gives you a better feeling for what the axioms do NOT allow to conclude. Namely, it is not possible to prove the boundedness of  $\mathbb{N}$  from the ordered field axioms alone.

**Example 2.5.x2:**  $\mathbb{R}(x)$  denotes the set of all rational functions of a real variable (defined on  $\mathbb{R} \setminus$  finitely many points). (Rational functions are defined to be quotients of polynomials). Addition and multiplication in the field are the usual addition and multiplication of functions. It is easy to see that the field axioms are satisfied. The constant functions 0 and 1 respectively are the neutrals for addition and multiplication. The set  $\mathbb{N}$  can be interpreted as a subset of  $\mathbb{R}(x)$ , by viewing the natural number  $n$  as the constant function  $n$ . Now  $\mathbb{N}$  can be defined from entirely within the field axioms by repeated addition of the 1-element of the field.

Now we define an order  $\prec$  on  $\mathbb{R}(x)$ . (There are many ways how we could do this, I am just giving one choice.) We say  $f \prec g$  iff there exists  $\varepsilon > 0$  such that  $f(x) < g(x)$  for all  $x \in ]0, \varepsilon[$  (the open interval that is denoted as  $(0, \varepsilon)$  by many other authors).

It is almost trivial to show that the transitivity, the addition and the multiplication axioms are verified in this situation. Checking the trichotomy axiom is a bit trickier: Suppose  $f \not\prec g$  and  $g \not\prec f$ . We must show that then  $f = g$ .

Now  $f \not\prec g$  means: For every  $\varepsilon > 0$ , there exists an  $x \in ]0, \varepsilon[$  such that  $f(x) \not< g(x)$ . Likewise  $g \not\prec f$  means: For every  $\varepsilon > 0$ , there exists an  $x \in ]0, \varepsilon[$  such that  $g(x) \not< f(x)$ . We can

then inductively construct a sequence  $x_1 > x_2 > x_3 > \dots > 0$  such that  $f(x_1) \leq g(x_1)$ ,  $f(x_2) \geq g(x_2)$ ,  $f(x_3) \leq g(x_3)$ ,  $f(x_4) \geq g(x_4)$ , etc with alternating inequality signs. By the intermediate value theorem, we can find a sequence  $\xi_i$  where  $\xi_i \in [x_{i+1}, x_i]$  such that  $f(\xi_i) = g(\xi_i)$ . Since there are infinitely many distinct  $\xi_i$  in this sequence, and since the equation  $f(\xi_i) = g(\xi_i)$  is equivalent to a polynomial equation by cross-multiplication, which has either only finitely many distinct solutions or else is identically fulfilled, it follows that  $f(x) = g(x)$  for all  $x$ , hence  $f = g$ .

[Remember, at this moment it is not crucial that you master every detail of this proof at a formal level, but rather that you understand the example at a pragmatic level, so that it can inform your intuition about the scope of the axioms.]

Now I claim that this ordered field does not satisfy the supremum axiom. The set  $\mathbb{N}$  (consisting of the constant functions whose value is a positive integer) is bounded above, because  $n < \frac{1}{x}$  for every  $n \in \mathbb{N}$ . For the same reason, the archimedean property is not satisfied: Given 1 and  $\frac{1}{x}$  in  $\mathbb{R}(x)$ , there is no natural number  $n$  such that  $n \cdot 1 > \frac{1}{x}$ . Therefore, from the proof of Thm. 2.5.12, this field cannot satisfy the supremum axiom. We can also see this directly: We use  $0 < 1$ , and  $0 < \frac{1}{2}$  (which is the multiplicative inverse of  $1 + 1$ ); these can be proven from the ordered field axioms (how?). Then we want to argue: If  $f$  is an upper bound for  $\mathbb{N}$ , then  $\frac{1}{2}f$  is also an upper bound for  $\mathbb{N}$ , and  $\frac{1}{2}f < f$ . So there is no smallest upper bound, i.e., no supremum of  $\mathbb{N}$ . Can you fill in the details? The trichotomy proof can be simplified; the sequence construction can be avoided.

### 3.1a: More examples of metric spaces

In all the examples of distances below, the symmetry and positivity are trivially satisfied; I will only comment on the triangle inequality, which is not trivial.

Refer to section 5.9 (Def 5.9.1, Prop. 5.9.3). You may replace the abstract notion ‘vector space’ with any of the specific examples  $\mathbb{R}^n$ ,  $C^0[a, b]$ . Given a norm  $\|\cdot\|$ , we obtain a metric by defining  $d(x, y) := \|x - y\|$ .

**Example 3.1.x1:** On  $\mathbb{R}^n$ , using the notation  $x = (x_1, \dots, x_n)$  for its elements  $x$ , we can define the euclidean distance  $d_2$  by  $d_2(x, y) := \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$ . This is a metric, coming from the norm  $\|x\|_2 := \sqrt{x_1^2 + \dots + x_n^2}$ . The non-trivial part is the triangle inequality, which can be proved from the Cauchy-Schwarz inequality in Linear Algebra. We’ll work out the details later (they are not difficult).

**Example 3.1.x2:** We can also introduce the taxi metric  $d_1$  on  $\mathbb{R}^n$ , which comes from a different norm,  $\|x\|_1 := |x_1| + \dots + |x_n|$ . The name comes from the idea that on a rectangular street grid, the road distance is given by the taxi distance, and is distinct from the euclidean distance (which is as the crow flies). The triangle inequality is an easy consequence of the triangle inequality in  $\mathbb{R}$ .

**Example 3.1.x3:** Another metric, called  $d_\infty$ , on  $\mathbb{R}^n$  comes from the supremum norm  $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$ . Proving the triangle inequality is an easy exercise.

**Exercise 3.1.X1:** In  $\mathbb{R}^2$ , plot the sets  $\{x \mid d(0, x) \leq 1\}$  for the metrics  $d_1$ ,  $d_2$ ,  $d_\infty$  respectively.

**Example 3.1.x4:** All these are special cases of the norm  $\|x\|_p := (|x_1|^p + \dots + |x_n|^p)^{1/p}$  for  $p \geq 1$ . Proof of the triangle inequality in this more general case will be postponed until a more convenient opportunity. (The same formula for  $p < 1$  would give a function  $\|\cdot\|_p$  that violates the triangle inequality.)

**Note:** Even for an analysis that is interested only in  $\mathbb{R}^n$ , the generality of the notion of metric space is useful. We will see later (and easily so) that notions of convergence and continuity can be expressed equivalently in terms of either of these metrics; the freedom to choose other metrics than the euclidean can help to give calculational simplicity in proofs.

**Example 3.1.x5:** This example does NOT come from a norm. Let  $S$  be the unit sphere in  $\mathbb{R}^3$ , i.e.,  $\{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ , and denote by  $\cdot$  the dot product in  $\mathbb{R}^3$ . Then the airplane distance is defined by  $d(x, y) := \arccos(x \cdot y)$ . We'll provide a quick proof of the triangle inequality below, for completeness. But the main focus here is that we have an example of a metric on a curved surface that is clearly a reasonable domain on which to study functions. So this example motivates why the generality of the notion 'metric' is useful.

FYI (and you are free to study or to skip this as preferred): Proof of the triangle inequality for the airplane distance: We use the dot and cross products in  $\mathbb{R}^3$ :

For vectors  $A, B, C \in \mathbb{R}^3$ , the following identity holds:

$$(A \cdot B)(C \cdot C) = (A \cdot C)(B \cdot C) + (A \times C) \cdot (B \times C)$$

It is straightforward, albeit lengthy, to prove this identity in terms of components. If we now assume  $\|A\|, \|B\|, \|C\| = 1$  and let  $a := d(B, C)$ ,  $b := d(A, C)$ ,  $c := d(A, B)$ , this formula becomes

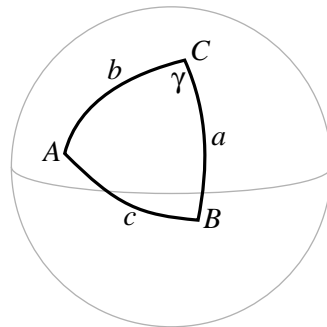
$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$

where  $\gamma$  is the angle between  $B \times C$  and  $A \times C$ . This is called the law of side cosines in spherical trigonometry. (In this context,  $A, B, C$  are vertices of a spherical triangle,  $a, b, c$  are its sides, and  $\gamma$  is the angle at  $C$ ). Now, since  $a, b \in [0, \pi]$  (and so their sines are nonnegative), we conclude

$$\cos c \geq \cos a \cos b - \sin a \sin b = \cos(a + b).$$

If  $a + b \leq \pi$ , we conclude by applying the arccos that  $c \leq a + b$ .

If  $a + b > \pi$ , the inequality  $c \leq a + b$  is trivial.



**Example 3.1.x6:** (This example is interesting in its own right, to get a feel for the generality of the notion of metric, but plays no significant role in calculus. It is a good source for counterexamples to seemingly plausible, but false conjectures.) Choose a prime number  $p$  and define a metric on the set  $\mathbb{Q}$  of rational numbers: Write  $x - y$  as a product of prime powers (e.g.,  $\frac{17}{6} - \frac{8}{9} = \frac{35}{18} = 2^{-1} \cdot 3^{-2} \cdot 5 \cdot 7$ ); in algebra/number theory it is shown that this factorization is unique; the  $p$ -adic distance  $d_p$  from  $x$  to  $y$  is the reciprocal of the power of  $p$  in this factorization. So  $d_2(\frac{17}{6}, \frac{8}{9}) = 2$ ,  $d_3(\frac{17}{6}, \frac{8}{9}) = 9$ ,  $d_5(\frac{17}{6}, \frac{8}{9}) = \frac{1}{5}$ ,  $d_{11}(\frac{17}{6}, \frac{8}{9}) = 1$ . [This  $d_p$  has nothing to do with the  $d_p$ 's on  $\mathbb{R}^n$  mentioned above.] The triangle inequality is satisfied even in the following stronger version:  $d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\} \leq d_p(x, y) + d_p(y, z)$ .

**Exercise 3.1.X2:** Show: If  $0 < u \leq v$ , then  $\frac{u}{1+u} \leq \frac{v}{1+v}$ , and use this property to show: If  $d$  is a metric, then  $d_* := d/(1+d)$  is a metric, too.

### 3.3a: lim sup and lim inf

(This material will be covered later – see pg 86. For now, just skim it as a sneak preview.)

**Exercise 3.3X1:** In any metric space, the following holds: The set of cluster points of a sequence is closed.

The trouble with proving convergence of a sequence in  $\mathbb{R}$  is often, that we want to do some calculations involving the presumptive limit of the sequence at a time when we have not proved convergence yet. But this calculation is still meant to be useful in actually proving convergence. The way out of this dilemma is provided by the notions of  $\limsup$  and  $\liminf$ .

**Definition 3.3.x1:** Let  $(x_n)$  be a sequence of real numbers. If  $(x_n)$  is NOT bounded above, we define  $\limsup x_n = \infty$ . Otherwise, we define  $\limsup x_n := \lim_{N \rightarrow \infty} \sup_{n \geq N} x_n$ . — Similarly, if  $(x_n)$  is NOT bounded below, we define  $\liminf x_n = -\infty$ . Otherwise, we define  $\liminf x_n := \lim_{N \rightarrow \infty} \inf_{n \geq N} x_n$ .

It is a corollary to Prop. 3.3.15 that the  $\limsup$  and  $\liminf$  of every real sequence exist. It is also immediate that  $\liminf x_n \leq \limsup x_n$

**Exercise 3.3.X2:** Show: If  $\lim x_n$  exists then  $\liminf x_n = \lim x_n = \limsup x_n$ . Conversely, if  $\limsup x_n = \liminf x_n$ , then  $\lim x_n$  exists.  $\limsup x_n$  and  $\liminf x_n$  are cluster points of the sequence  $(x_n)$ , and any cluster point  $s$  of the sequence  $x_n$  satisfies  $\liminf x_n \leq s \leq \limsup x_n$ .

The following neat exercise is actually a lemma that is useful in dynamical systems and in ergodic theory. Its proof illustrates the idea to calculate with  $\liminf$  and  $\limsup$  in order to prove later, based on these calculations, that the limit exists.

**Exercise 3.3.X3:** Let  $(x_n)$  be a sequence of real numbers that is *subadditive*, i.e.,  $x_{k+l} \leq x_k + x_l$  for all  $k, l \in \mathbb{N}$ . If  $(x_n)$  is also bounded below, then  $\lim \frac{x_n}{n}$  exists. — Hint: For each  $m, r \in \mathbb{N}$ , consider the sequence  $(x_{km+r}/(km+r))_k$  and show that  $\limsup_{k \rightarrow \infty} \frac{x_{km+r}}{km+r} \leq \frac{x_m}{m}$ . Next conclude that  $\limsup_{n \rightarrow \infty} \frac{x_n}{n} \leq \inf_m \frac{x_m}{m}$ . Finally, conclude the proof. Re-examine your proof. The hypothesis that  $x_n$  be bounded below has not entered in full strength. What weaker hypothesis was actually used?

### 3.4a: Semicontinuity

**Definition 3.4.x1:** A function  $f : X \rightarrow \mathbb{R}$  is called *lower semicontinuous (lsc)* at  $x$ , if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $f(y) \geq f(x) - \varepsilon$  for all  $y \in B(x, \delta)$ . It is called *upper semicontinuous (usc)* at  $x$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $f(y) \leq f(x) + \varepsilon$  for all  $y \in B(x, \delta)$ .

This definition is occasionally useful. We'll find it helpful during our approach to integration theory. Lower semi-continuity plays a crucial role in minimization problems in calculus of variations, where we ask for the minimum of a function  $f : X \rightarrow \mathbb{R}$  in situations where  $X$  itself is a set consisting of real functions (single or multi-variable).

**Exercise 3.4.X1:** Show:

- (1)  $f : X \rightarrow \mathbb{R}$  is continuous if and only if it is both upper and lower semicontinuous.
- (2) Accepting the properties of the sine from elementary calculus, show that  $f(x) := \sin 1/x$  for  $x \neq 0$  and  $f(0) := a$  defines a function that is lsc at  $x = 0$  iff  $a \leq -1$ , and that is usc at  $x = 0$  iff  $a \geq 1$ .
- (3) Show that  $f : X \rightarrow \mathbb{R}$  is usc iff  $f^{-1}(]-\infty, c])$  is open for every  $c \in \mathbb{R}$ , that  $f : X \rightarrow \mathbb{R}$  is lsc iff  $f^{-1}(]c, \infty[)$  is open for every  $c \in \mathbb{R}$ , and that  $f : X \rightarrow \mathbb{R}$  is continuous iff  $f^{-1}(]a, b])$  is open for every  $a, b \in \mathbb{R}$ .
- (4) Show that  $f$  is usc at  $x$  iff  $\lim_{\delta \rightarrow 0} \sup_{B(x, \delta)} f \leq f(x)$ . State an analogous statement for lsc.

### 3.5a: Cover compactness

First observe the remark below definition 3.5.1 and Rmk 3.5.13 in the printed notes: There are two definitions of compactness that are equivalent in metric spaces but become distinct in more general settings. The one presented in the notes according to Def 3.5.1 is usually called ‘sequentially compact’. The other definition (showing up as a conclusion in Prop 3.5.14) is usually just called compact, but may be called ‘cover compact’ for underlining the distinction more clearly.

In addition to the material provided by the book, I’d like to include a full equivalence proof ‘cover compact’  $\iff$  ‘sequentially compact’ in the course. To this end, it may help if you clarify the distinction by replacing the word ‘compact’ with ‘sequentially compact’ in the book all the way up to Exercise 3.37.

Here I will create a clone of the first part of the book’s chapter 3.5, reproving all theorems in terms of ‘cover compact’. This should give you the skill to handle the notion of open covers in practical settings. The cloned theorems will have the corresponding numbers, with a ‘C’ for cover attached.

Read Def 3.5.12.

**Definition 3.5.1C:** *A subset  $A$  of a metric space  $X$  is called cover compact if every open cover  $\{U_\lambda\}_{\lambda \in \Lambda}$  of  $A$  has a finite subcover, i.e., a finite subcollection  $\{A_{\lambda_1}, \dots, A_{\lambda_k}\}$  that covers  $A$ .*

**Proposition 3.5.2C:** *If  $A$  is a cover compact subset of a metric space  $A$ , then  $A$  is closed and bounded.*

**Proof:** There is nothing to prove if  $X = \emptyset$ . For boundedness, we choose  $\Lambda = \mathbb{N}$  and one point  $x_0 \in X$ , and we let  $U_\lambda := B(x_0, \lambda)$ . These provide an open cover for all of  $X$  and therefore for  $A$  also. Since  $A$  is cover compact, there is a finite subcover  $\{U_\lambda \mid \lambda \in \{n_1, \dots, n_k\}\}$ . Without loss of generality, let  $n_k$  be the maximum of  $\{n_1, \dots, n_k\}$ . Then  $A \subset B(x_0, n_k)$ , hence is bounded.

To show that  $A$  is closed, consider  $x \in \bar{A}$ . We want to show  $x \in A$ . To this end, we assume  $x \notin A$ , and from this we construct an open cover of  $A$  that has no finite subcover. Letting  $\Lambda := \mathbb{N}$  again, we define  $U_n := X \setminus C(x, \frac{1}{n})$ , the complement of a closed ball. Since  $\bigcap C(x, \frac{1}{n}) = \{x\}$ , we have  $\bigcup U_n = X \setminus \{x\} \supset A$ . For any finite subcollection of  $\{U_n\}$ , the union would be  $X \setminus C(x, \frac{1}{n_{max}})$ . This does not contain  $A$ , because  $C(x, \frac{1}{n_{max}}) \cap A \supset B(x, \frac{1}{n_{max}}) \cap A \neq \emptyset$ . ■

**Prop. 3.5.3C:** *If  $A$  is a cover compact subset of a metric space  $X$  and  $B$  is a closed subset of  $A$ , then  $B$  is cover compact.*

**Proof:** Let  $\{U_\lambda \mid \lambda \in \Lambda\}$  be an open cover of  $B$ . Then  $\{U_\lambda \mid \lambda \in \Lambda\} \cup \{X \setminus B\}$  is an open cover of  $X$  and therefore of  $A$ . Take a finite subcollection covering  $A$ . It is no loss of generality to assume that  $X \setminus B$  is among this finite subcover, for otherwise we just join it. So we have this finite cover of  $B$  trivially:  $U_{\lambda_1} \cup \dots \cup U_{\lambda_k} \cup (X \setminus B) \supset A \supset B$ . But then  $U_{\lambda_1} \cup \dots \cup U_{\lambda_k} \supset B$  as well, because  $X \setminus B$  is disjoint from  $B$ . ■

**Exercise 3.36C:** Show that any singleton set  $\{x\}$  is cover compact.

Answer (with the usual notation): If  $\{x\} \subset \bigcup U_\lambda$ , then  $x \in \bigcup U_\lambda$ ; hence  $x \in U_{\lambda_0}$  for some  $\lambda_0 \in \Lambda$ , so  $\{x\}$  is covered by a single set, which is a finite subcover.

**Note:** Here and in the following we freely use results like  $\lim 1/(2^n) \rightarrow 0$ . This follows from  $2^i > i$ , which in turn is proved by induction, and the sandwich theorem (Exercise 3.20).

**Thm 3.5.5C:** (Heine-Borel) *A subset of the reals is cover compact if and only if it is bounded and closed.*

**Proof:** The ‘ $\implies$ ’ part is Prop. 3.5.2C. For the converse, assume  $A$  is closed and bounded. Letting  $a := \inf A = \min A$  and  $b := \sup A = \max A$ , we have  $A \subset [a, b]$ . In view of Prop 3.5.3C, it suffices to show that  $[a, b]$  is cover compact. We provide an indirect proof, assuming  $\{U_\lambda\}$  is an open cover of  $[a, b]$  for which there is no finite subcover. Let  $a_0 := a$  and  $b_0 := b$  and construct a sequence of intervals  $[a_i, b_i]$  inductively by successive bisection. Namely, assuming that  $[a_i, b_i]$  has been constructed in such a way that no finite subcollection of  $\{U_\lambda\}$  covers  $[a_i, b_i]$ , we notice that at least one of the following is true: No finite subcollection covers  $[a_i, \frac{1}{2}(a_i + b_i)]$ , or no finite subcollection covers  $[\frac{1}{2}(a_i + b_i), b_i]$ . Accordingly we define  $[a_{i+1}, b_{i+1}]$  as one of these two intervals. Now we have a sequence of nested intervals  $[a_i, b_i] \supset [a_{i+1}, b_{i+1}]$ . The increasing sequence  $(a_i)$ , which is bounded above by  $b_0$ , has a limit, which we call  $x_*$ . The decreasing sequence  $(b_i)$ , which is bounded below by  $a_0$ , also has a limit, which we call  $x^*$ . But by construction  $b_i - a_i = (b_0 - a_0)/2^i \rightarrow 0$ , so  $x_* = x^*$ . Since  $x_* \in [a, b]$ , there must be some  $\lambda$  such that  $U_\lambda \ni x_*$ . Since  $U_\lambda$  is open, there exists  $\varepsilon > 0$  such that  $]x_* - \varepsilon, x_* + \varepsilon[ \subset U_\lambda$ . But as soon as  $(b_0 - a_0)/2^i < \varepsilon$ , which is the case for  $i$  sufficiently large, we will have  $[a_i, b_i] \subset ]x_* - \varepsilon, x_* + \varepsilon[$  (using  $x_* \in [a_i, b_i]$ ). But this contradicts the construction, according to which  $[a_i, b_i]$  cannot be covered by finitely many of the  $U_\lambda$ , specifically not by a single one. ■

**Thm 3.5.6C:** *Let  $X$  and  $Y$  be metric spaces,  $A$  be a cover compact subset of  $X$ , and  $f : X \rightarrow Y$  continuous. Then  $f(A)$  is a cover compact subset of  $Y$ .*

**Proof:** Let  $\{U_\lambda\}$  be an open cover of  $f(A)$ . Then  $\{V_\lambda := f^{-1}(U_\lambda)\}$  is a collection of open sets by Prop. 3.4.8, and it covers  $f^{-1}(f(A))$ . [Can you fill in the details here? Let me just do it one more time, to play it safe. If  $x \in f^{-1}(f(A))$ , this means that  $f(x) \in f(A)$  by definition of inverse image. Since  $\bigcup U_\lambda \supset f(A) \ni f(x)$ , there is some  $\lambda$  such that  $f(x) \in U_\lambda$ . But this means that  $x \in f^{-1}(U_\lambda) = V_\lambda$ .]

Since  $A \subset f^{-1}(f(A))$ , the  $V_\lambda$  cover  $A$ , and since  $A$  is compact, there is a finite subcover  $\{V_{\lambda_1}, \dots, V_{\lambda_k}\}$  for  $A$ . Then the corresponding  $\{U_{\lambda_1}, \dots, U_{\lambda_k}\}$  covers  $f(A)$ . ■

**Thm 3.5.9C:** *Let  $\{A_i\}_{i=1}^\infty$  be a collection of non-empty cover compact subsets of a metric space  $X$  that is nested, in the sense that if  $i < j$  then  $A_j \subset A_i$ . Then  $\bigcap_{i=1}^\infty A_i \neq \emptyset$ .*

**Proof:** Assuming the intersection to be empty, we construct an open cover of  $A_1$  that has no finite subcover. Namely we let  $U_i := X \setminus A_i$ , which are open sets, because the (cover compact)  $A_i$  are closed. The collection  $\{U_i\}$  covers all of  $X$  when  $\bigcap A_i = \emptyset$ . So in particular it covers  $A_1$ . If we consider any finite subcover  $U_{i_1} \cup \dots \cup U_{i_k}$  (which equals  $U_{i_k} = X \setminus A_{i_k}$  if  $i_k$  is the largest of the subscripts), and select some  $x \in A_{i_k} \subset A_1$  (since the  $A$  are non-empty), then we observe that  $U_{i_k} = X \setminus A_{i_k} \not\ni x$ . So this finite collection fails to cover  $A$ . ■

After developing cover-compactness in analogy to sequential compactness, we now prove the equivalence of the two notions in metric spaces.

**Theorem 3.5.14C(1):** *If a metric space  $X$  is sequentially compact, then it is cover compact.*

**Proof:** The case  $X = \emptyset$  being trivial, we now assume  $X \neq \emptyset$ . Let  $\{U_\lambda\}$  be an open cover of  $X$ . By proposition 3.5.14, there exists  $\varepsilon > 0$  such that for every  $x \in X$ , there exists some  $\lambda = \lambda(x)$  such that the ball  $B(x, \varepsilon)$  is entirely contained in  $U_\lambda$ . It suffices to show that the open cover  $\mathcal{B} := \{B(x, \varepsilon)\}_{x \in X}$  has a finite subcover  $\{B(x_1, \varepsilon), \dots, B(x_n, \varepsilon)\}$ . For in this case,  $\{U_{\lambda(x_1)}, \dots, U_{\lambda(x_n)}\}$  will be a finite subcover of the original cover.

Assume  $\mathcal{B}$  does not have a finite subcover. We will construct a sequence that cannot have a convergent subsequence (and hence not a cluster point), in violation of sequential compactness. To this end choose  $y_1 \in X$  arbitrarily. Assuming we have  $y_1, \dots, y_n$  chosen already, we choose  $y_{n+1} \in X \setminus \bigcup_{i=1}^n B(y_i, \varepsilon)$ . This latter set is not empty, because we assumed  $X$  is not covered by any finite subcover of  $\mathcal{B}$ . By construction  $d(y_i, y_j) \geq \varepsilon$  for  $j > i$ . And by symmetry the same holds for  $j \neq i$ . If the sequence  $(y_i)$  had a convergent subsequence, whose limit may be called  $y_*$ , then infinitely many of the  $y_i$  would have to be within  $B(y_*, \frac{1}{2}\varepsilon)$ , but this contradicts the fact that any two of the  $y_i$  have distance  $\geq \varepsilon$ . ■

**Theorem 3.5.14C(2):** *If a metric space  $X$  is cover compact, then it is sequentially compact.*

**Proof:** by contrapositive. Suppose  $(x_n)_{n=1}^\infty$  is a sequence without cluster point. We construct an open cover of  $X$  that has no finite subcover. The set  $A := \{x_n\}$  is closed by Prop. 3.3.12 (because the sequence has no cluster point), so  $U_0 := X \setminus A$  is open. Since the sequence  $(x_n)$  has no cluster point, in particular  $x_1$  is not a cluster point, so there exists  $\varepsilon_1 > 0$  such that  $B(x_1, \varepsilon_1)$  contains at most finitely many terms of the sequence. Disregarding those that may be equal to  $x_1$ , we can decrease  $\varepsilon_1$  to make sure that  $B(x_1, \varepsilon_1)$  contains no other terms of the sequence (except possible repeats, of which there are at most finitely many). In other words  $B(x_1, \varepsilon_1)$  is disjoint from the set  $A \setminus \{x_1\}$ . Inductively, once we have constructed  $x_1, \dots, x_n$  and  $\varepsilon_1 \geq \dots \geq \varepsilon_n$ , we find  $x_{n+1}$  and  $\varepsilon_{n+1} \leq \varepsilon_n$  such that  $B(x_{n+1}, \varepsilon_{n+1})$  is disjoint from  $A \setminus \{x_1, \dots, x_n\}$ . Now the balls  $U_n := B(x_n, \frac{1}{2}\varepsilon_n)$  are pairwise disjoint, because  $y \in B(x_n, \frac{1}{2}\varepsilon_n) \cap B(x_m, \frac{1}{2}\varepsilon_m)$  (with  $m > n$ ) would imply  $d(x_n, x_m) \leq d(x_n, y) + d(y, x_m) < \frac{1}{2}\varepsilon_n + \frac{1}{2}\varepsilon_m \leq \varepsilon_n$ , so  $x_m$  would be in  $B(x_n, \varepsilon_n)$ , contrary to the construction.

Now  $\bigcup_{n=0}^\infty U_n = X$ , but no finite subcollection (without loss of generality  $\{U_n\}_{n=0}^N$ ) can cover  $X$ , specifically it does not cover  $x_{N+1}$ . ■

### 3.5b: Cantor sets

An example of a compact set in  $\mathbb{R}$ , in two variants. Every analyst needs to know this example, which is good in killing many a too-naive conjecture.

**Example 3.5.x1:** Let  $A_0 := [0, 1]$ . Let  $A_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Inductively define  $A_n$  as the union of  $2^n$  many closed intervals, each of length  $3^{-n}$ , obtained by removing the middle third from each of the intervals of which  $A_{n-1}$  is made up. The standard Cantor set is defined as  $C := \bigcap A_n$ .

As an intersection of a nested sequence of compact non-empty sets, it is non-empty and compact (by Thm 3.5.9). In the spirit of Cor 3.5.10 (with a few details pertaining to the next sections on subspaces skipped here), it can be shown that  $C$  is uncountable.

While we have yet to define rigorously a notion of measure (generalizing the notion of length of an interval in  $\mathbb{R}$ ), let us just notice informally with the purpose of motivating a future definition of length ('measure'), that any sensible length we may assign to  $C$  ought to be less

that  $2^n/3^n$ , because  $C \subset A_n$ . So, being nonnegative by another reasonable stipulation, the length ('measure') of  $C$  ought to be (and will be) 0.

$C$  is 'full of holes' in the following sense: Between any two points  $x, y \in C$  with  $x < y$ , there exists a point  $z$  that is NOT in  $C$ . In particular,  $\overset{\circ}{C} = \emptyset$ .

**Example 3.5.x2:** 'Fat Cantor set'. This is a variant of the same construction. The fat Cantor set will share all properties of the Cantor set that were mentioned above, except that it 'ought to have' (and will have, once rigorously defined) positive length (positive measure). Again we start with  $A_0 = [0, 1]$ . But now  $A_n$  is designed to have length  $\frac{1}{2} + 2^{-n-1}$ . To this end, remove an interval of length  $\frac{1}{4}$  from the middle of  $A_0$  to get  $A_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$ . Now remove two intervals of total length  $\frac{1}{8}$  (namely length  $\frac{1}{16}$  each) from the middle of each of the two intervals that make up  $A_1$ . This gives us  $A_2 = [0, \frac{5}{32}] \cup [\frac{7}{32}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{25}{32}] \cup [\frac{27}{32}, 1]$ . Continuing this way, let  $C_{\text{fat}} := \bigcap A_n$ .

The length of  $C_{\text{fat}}$  ought to be (and will be defined as)  $\lim(\frac{1}{2} + 2^{-n-1}) = \frac{1}{2}$ .

**Addendum to Example 3.5.11:** In the space  $X := C^0[0, 1]$  of continuous real valued functions on  $[0, 1]$ , with the sup distance, here is an example of a nested family  $A_n$  of bounded and closed sets, with empty intersection. Note that in this metric space, bounded and closed does not imply compact. (At least we haven't proved any theorem to this effect, and this example will now show that such an implication can indeed not hold.)

Take  $A_n := \{f \in C^0[0, 1] \mid f(0) = 0 \text{ and } \frac{nx}{1+nx} \leq f(x) \leq 1 \text{ for all } x \in ]0, 1]\}$ .

The sets  $A_n$  are clearly bounded, since they lie in the ball  $B(\text{zerofunction}, 1.001)$ . The nesting of the sequence is trivial. To see that the sets  $A_n$  are closed, we may notice that they are defined by pointwise non-strict inequalities ( $\leq$ ), and these automatically define *closed* sets in  $C^0[0, 1]$ . Indeed, consider, for each  $x \in ]0, 1]$ , the function  $\text{ev}_x : C^0[0, 1] \rightarrow \mathbb{R}$ , defined by  $\text{ev}_x(f) := f(x)$ . These are called the evaluation functions. They are continuous, with  $\delta = \varepsilon$ , because  $|f(x) - g(x)| \leq d(f, g)$ . *If you find this confusing, make sure you distinguish carefully between  $f \in C^0[0, 1]$ , which is a function, and its values  $f(x) \in \mathbb{R}$ , which are real numbers, and the function  $\text{ev}_x$ , which assigns to the function  $f$  its value at a given  $x$ .*

So  $A_n = \text{ev}_0^{-1}(\{0\}) \cap \bigcap_{x \in ]0, 1]} \text{ev}_x^{-1}([\frac{nx}{1+nx}, 1])$ . Inverse images of closed sets under continuous functions are closed, and therefore  $A_n$  is closed as an intersection of closed sets.

Now any continuous function  $f \in \bigcap_n A_n$  must satisfy  $f(0) = 0$  and  $\frac{nx}{1+nx} \leq f(x) \leq 1$  for every  $x > 0$  and every  $n \in \mathbb{N}$ . But since  $\sup_n \frac{nx}{1+nx} = \lim_{n \rightarrow \infty} \frac{nx}{1+nx} = 1$  for every  $x > 0$ , this implies  $f(x) = 1$  for  $x > 0$ , whereas  $f(0) = 0$ . This contradicts continuity of  $f$ .