# Math 445, Fall 2016: — Homework

**Hwk #1:** (from Sec. 1.1, #17 — typo in book's part (c): $b^2 < c$, not $r$)

For $c > 0$, define $S := \left\{ x \in \mathbb{R} \;\middle|\; x \geq 0\,,\ x^2 < c \right\}$.
(a) Show that $S$ is bounded above and nonempty (eg., that $c+1$ is an upper bound). Then using the supremum axiom (completeness axiom), let $b := \sup S$.
(b) Assuming $b^2 > c$, find some small $r > 0$ such that $b - r$ is still an upper bound for $S$, thus contradicting that $b$ is the *least* upper bound.
(c) Assuming $b^2 < c$, find some small number $r > 0$ such that $b + r$ is still in $S$, thus contradicting the choice of $b$ being an upper bound at all.
(d) Conclude that $b^2 = c$.
(e) Prove that if $b_1^2 = c$ and $b_2^2 = c$ and $b_1, b_2 > 0$, then $b_1 = b_2$, so there is a *unique* number $b > 0$ such that $b^2 = c$. We call this number $\sqrt{c}$.

**Hwk #2:** (from Sec. 1.3, #13)

Prove by induction that $(1+x)^n \geq 1 + nx$ for $n \in \mathbb{N}$ and $x > -1$.

**Hwk #3:** (from Sec. 1.3, #14–16)

(a) Show for any $a, b \in \mathbb{R}$ that $ab \leq \frac{1}{2}(a^2 + b^2)$.
(b) Conclude for $a, b \geq 0$ that $\sqrt{ab} \leq \frac{1}{2}(a+b)$. (The inequality of the geometric and arithmetic mean, short adm–inequality)
(c) Prove for arbitrary $\varepsilon > 0$ and $a, b \in \mathbb{R}$ that $ab \leq \frac{1}{2}(\varepsilon a^2 + b^2/\varepsilon)$.

**Hwk #4:** (Sec. 2.1, #1)

≪ see book ≫

**Hwk #5:** (Sec. 2.1, #3a)

≪ see book ≫

**Hwk #6:** (Sec. 2.1, #10)

≪ see book ≫

**Hwk #7:** (Sec. 2.1, #11)

$\ll$ see book $\gg$. Also characterize what each property does mean, if it's not equivalent to convergence.

**Hwk #8:** (Sec. 2.1, #15 & #17)

$\ll$ see book $\gg$.

**Hwk #9:** (Sec. 2.1, #18)

$\ll$ see book $\gg$. *Hint: First define a $k$ such that for $n > k$, $|a_n - a| < \varepsilon/2$. Then estimate $\sigma_n$ for $n > N > k$ with appropriately chosen $N$.*

**Hwk #10:** (Sec. 2.2, #3)

$\ll$ see book $\gg$.

**Hwk #11:** (Sec. 2.2, #5)

$\ll$ see book $\gg$.

**Hwk #12:** (Sec. 2.3, #2)

$\ll$ see book $\gg$.

**Hwk #13:** (Sec. 2.3, #11)

$\ll$ see book $\gg$. (The referenced sec 2.3 #10 was also our number 3(b)) (To build experience, use a pocket calculator to find the limit in question for $a = 10$ and $b = 1$, to 8 decimals.)

**Hwk #14:**

In this problem, we will define $\exp(x) := \lim_{n\to\infty}(1+\frac{x}{n})^n$ and prove from it the properties of the exponential function commonly called $e^x$. DO NOT USE PROPERTIES OF THE NUMBER $e$ OR THE FUNCTION $x \mapsto e^x$ YET. THEY ARE TO BE PROVED FROM THIS DEFINITION.
(a) Show that for $x > -1$, the sequence $(a_n)$ given by $a_n = (1+\frac{x}{n})^n$ is increasing. *Hint: Write* $(1+\frac{x}{n+1})^{n+1}/(1+\frac{x}{n})^{n+1}$ *as* $(1-?)^{n+1}$ *and use hwk. #2* — Note that even for arbitrary $x$, the same is true for the subsequence $(a_n)_{n>|x|}$.
(b) Writing now $a_n(x)$ instead of $a_n$ to have the dependence on $x$ explicit, show that $a_n(x)a_n(-x) < 1$ for large enough $n$, and conclude that $a_n(x)$ is bounded above. Using Hwk. #2 again, also conclude that $a_n(x)a_n(-x)$ converges to 1 for every $x$.
(c) Letting $\exp(x) := \lim_{n\to\infty} a_n(x)$, show that $\exp(x)\exp(-x) = 1$. Also, if $x$ and $y$ have the *same* sign ($xy \geq 0$), prove that $a_n(x)a_n(y) \geq a_n(x+y)$ and conclude the analog for $\exp(x)$. Taking reciprocals, you should be able to prove $\exp(x)\exp(y) = \exp(x+y)$ provided $xy \geq 0$.
(d) How would you prove $\exp(x)\exp(y) = \exp(x+y)$ when $x$ and $y$ have opposite sign (or $xy \leq 0$) ?

**Hwk #15:** (Sec. 2.4, #1)

$\ll$ see book $\gg$.

**Hwk #16:** (Sec. 2.4, #11)

$\ll$ see book $\gg$.

**Hwk #17:** (Sec. 2.4, #12)

$\ll$ see book $\gg$.

**Hwk #18:** (Sec. 2.5, #1)

$\ll$ see book $\gg$.

**Hwk #19:** (Sec. 2.5, #7)

$\ll$ see book $\gg$.

**Hwk #20:** (Sec. 2.5, #8)

$\ll$ see book $\gg$.

**Hwk #21:**

(a) Composition of continuous functions is continuous: Read the book page 56-57 for this – nothing to grade here; we need the material, but I won't use lecture time on this one.
(b) The square root function is continuous: Prove this. *Hint: Assume* $x_n \to x_*$ *(with* $x_n, x_* \geq 0$*). Assume for contradiction that* $\sqrt{x_n} \not\to \sqrt{x_*}$*. Find a subsequence converging elsewhere. Draw a conclusion based on the continuity of the square function.*

**Hwk #22:**

(Sec. 3.1, #3 and #6) $\ll$ see book $\gg$.

**Hwk #23:**

(Sec. 3.1, #9) $\ll$ see book $\gg$.

**Hwk #24:** (Sec. 3.1, #14)

$\ll$ see book $\gg$.

**Hwk #25:** (Sec. 3.2, #2)

$\ll$ see book $\gg$.

**Hwk #26:** (Sec. 3.3, #3)

$\ll$ see book $\gg$.

**Hwk #27:** (Sec. 3.3, #8 and #9)

$\ll$ see book $\gg$.

**Hwk #28:**

Prove in two ways that the square root function $\sqrt{\cdot} : [0, \infty[ \to [0, \infty[$ is uniformly continuous: by the sequence definition, and using $\varepsilon$ and $\delta$.

**Hwk #29:** (Sec. 3.4, #7)

$\ll$ see book $\gg$.

**Hwk #30:** (Sec. 3.5, #5)

$\ll$ see book $\gg$.

**Hwk #31:** (Sec. 3.5, #9)

$\ll$ see book $\gg$.

**Hwk #32:**

Prove that $\lim_{x \to 0} \frac{\exp x - 1}{x} = 1$. *Hint: Recall the estimates I used in class to prove* exp *is continuous at 0.* Then prove that $\lim_{y \to x_0} \frac{\exp y - \exp x_0}{y - x_0} = \exp x_0$.

**Hwk #33:** (from Sec. 4.3, #6, with some extra info by me)

Prove that the equation $x^4 + 2x^2 - 6x + 2 = 0$ has exactly two solutions. *Hint: use both IVT and monotonicity of $f'$.* State the theorems that you are using.

**Hwk #34:**

For real numbers $a, b$, show that the equation $x^3 + ax + b$ has exactly 3 real solutions if and only if $4a^3 + 27b^2 < 0$. *Hint: Determine the position of the local extrema.*

**Hwk #35:**

Consider the following possible strengthening of the Cauchy MVT:
¿"If $f$ and $g$ are differentiable in $]a,b[$ and continuous on $[a,b]$ and $g(b) \neq g(a)$, then there exists $c \in ]a,b[$ such that $g'(c) \neq 0$ and $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'c)}$."?
I had written this down tentatively in class, and proved $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$, but then was not able to divide by $g'(c)$ so I amended the thm to require $g' \neq 0$.

Show that the conjecture is not true by considering $f(x) = x^2 + \frac{1}{3}x^3$ and $g(x) = x^2 - \frac{1}{3}x^3$ on $[-1, 1]$.

**Hwk #36:**

For the function $f$ given by $f(x) = 4x(1 - x)$ on the interval $[0, 1]$, take an equidistant partition $P$ with 5 intervals. Calculate numerically the upper and lower Darboux sums $U(f, P)$ and $L(f, P)$.

**Hwk #37:** (Sec. 6.1, # 6)

**Hwk #38:** (Sec. 6.3, # 1)

Suppose that the functions $f$, $g$, $f^2$, $g^2$, and $fg$ are integrable over the compact interval $[a, b]$. Prove that $(f - g)^2$ is also integrable over $[a, b]$, and that $\int_a^b (f - g)^2 \, dx$. Use this to prove that

$$\int_a^b f(x)g(x)\, dx \le \frac{1}{2}\left[\int_a^b f^2(x)\, dx + \int_a^b g^2(x)\, dx\right] .$$

**Hwk #39:** (Sec 6.3., #2)

Cauchy–Schwarz Inequality for Integrals: Suppose that the functions $f$, $g$, $f^2$, $g^2$, and $fg$ are integrable over $[a, b]$. Prove that

$$\int_a^b f(x)g(x)\, dx \le \left(\int_a^b f(x)^2\, dx\right)^{1/2} \left(\int_a^b g(x)^2\, dx\right)^{1/2} .$$

**Hint:** For each number $\lambda$, define $p(\lambda) := \int_a^b (f(x) - \lambda g(x))^2 \, dx$. Show that $p(\lambda)$ is a quadratic polynomial that is nonnegative for all $\lambda \in \mathbb{R}$; therefore its discriminant is not positive.

**Hwk #40:** (Sec 6.4., #3)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and that $\int_a^b f(x)\, dx = 0$. Prove that there is some point $x_0$ in $[a, b]$ for which $f(x_0) = 0$. ≪Hint given in book.≫

**Hwk #41:** (Sec 6.4., #5)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and that the inequality $\int_c^d f(x)\, dx \le 0$ holds for any $c, d$ satisfying $a \le c < d \le b$. Prove that $f(x) \le 0$ for all $x \in [a, b]$. — Is the same conclusion still valid if we only assume integrability of $f$ instead of continuity?

**Hwk #42:** (Sec 6.5., #5 – slightly modified and extended)

For this example, you may use knowledge of the sine and cosine function, in particular that $\sin' = \cos$ and $\cos' = -\sin$; and that $\cos x \le 1$ for all $x \in \mathbb{R}$. [You won't need any further properties of these functions.]

By repeated use of the monotonicity property of the integral, show that each of the following inequalities (each for $x \ge 0$) implies the next one:

$$\begin{aligned} \cos x &\le 1 \\ \sin x &\le x \\ \cos x &\ge 1 - \tfrac{x^2}{2} \\ \sin x &\ge x - \tfrac{x^3}{6} \end{aligned}$$

Keep going and prove two more inequalities (one for cosine, and one for sine) in the same manner. Where does the hypothesis $x \ge 0$ enter into the argument?