# Homework
# UTK – M431 – Differential Equations
## Spring 2003, Jochen Denzler, MWF 11:15–12:05, BU 475

**7.** See notes on the Frobenius method, page 8

**8.** See notes on the Frobenius method, page 8

**9.** Evaluate the Bessel function $J_2(x)$ with a pocket calculator with a precision of 4 digits behind the decimal point. Record how many terms you needed for the required precision. We want $x = 2.7, 2.8, \ldots, 3.1, 3.2$ and $x = 1.3, 1.4, \ldots 1.8$. Each of you should select one of each set at random and e-mail me the results for hwk credit.

**10.** In the following problems, I want you not only to find the recursion formula (and possibly a closed formula for the power series coefficients), but also to trace back the highest index in the recursion formula to deriviatives in the ODE. For example, if you come up with a recursion formula $na_n = a_{n-1} - \frac{2}{n}a_{n-2}$, the highest index term is $a_n$, and the question is: which of the terms $\ldots \times y''$, $\ldots \times y'$, $\ldots \times y$ has/have contributed to the $na_n$ in the recursion formula. Answer this question for the hwk problems 6a–d and the Bessel equation now. You will observe a pattern that is generally true and will illuminate the distinction between regular, regular singular, and irregular singular points.

| ODE | type of point | highest coeff in recursion comes from<br>highest deriv' only | <br>highest and other deriv's (specify) | <br>only lower deriv's (specify) |
|---|---|---|---|---|
| Hwk 6a | regular | | | |
| Hwk 6b | regular | | | |
| Hwk 6c | regular | | | |
| Hwk 6d | regular | | | |
| Bessel eqn | reg' singular | | | |
| hypergeometric | $x = 0$ reg' singular | | | |
| hypergeometric | $x = 1$ reg' singular | | | |
| | | | | |
| | | | | |

The final entries in the table refer to problems below.

**11.** The hypergeometric ODE has three real (or, possibly, complex) numbers $\alpha$, $\beta$, $\gamma$ as parameters. If you need to make choices, distinguish cases, or assume conditions on these parameters in subsequent calculations, make the choice that is appropriate for $\alpha = 2.798137$, $\beta = 0.318564$, $\gamma = -1.337921$. But you are advised to retain the parameters $\alpha$, $\beta$, $\gamma$, if only as “abbreviations”. The hypergeometric equation is

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0 \ .$$

Find two linearly independent solutions of this equation by means of the power series method (aka Frobenius method), centered at $x = 0$. What are the characteristic exponents of the regular singular point $x = 0$? Make sure that you get a general (closed) formula for the coefficients $a_n$. If you find this difficult, and you have come up with an expression ($n^2$+?$n$+?) somewhere in the recursion formula, make sure that you factor this expression, because that will clarify the general pattern.

**12.** Show that $x = 1$ is also a regular singular point of the hypergeometric equation (see previous problem). Find two linearly independent solutions in the form of power series centered at $x = 1$. What are the characteristic exponents of $x = 1$?

**13.** Let $y = y(x)$ be a solution of the hypergeometric ODE. Let $z = 1/x$ and write $u(z) = y(1/z) = y(x)$. Using the chain rule, translate the ODE for $y$ into an ODE for $u$. What kind of point (regular, regular singular, irregular singular) is $z = 0$ for this new ODE?

Explanation of purpose: We call infinity a regular / regular singular / irregular singular point of an ODE for $y = y(x)$, if $z = 0$ is a regular / regular singular / irregular singular point of the transformed equation for $u = u(z) = y(1/z)$. — Given a linear homogeneous ODE with analytic coefficients, the first thing to do is to find all singular points, real, complex, or infinity, and to classify them as regular singular (and find its characteristic exponents) or irregular singular. Together with some complex variable knowledge, this is information is almost like a finger print of the ODE and determines or hints in which chapter of the big books about special functions you will find information on the solutions to this ODE. We will not pursue this FBI (Function Behavior Identification) task further.

**14.** We have found only one (up to constant multiple) solution to the Bessel equation of index 0

$$x^2y'' + xy' + x^2y = 0$$

namely the function $J_0$ given in the course manuscript. To find another, linearly independent solution, try

$$\tilde{Y}_0(x) := (\ln x)J_0(x) + \sum_{n=0}^{\infty} \tilde{a}_n x^n$$

and determine the $\tilde{a}_n$. Note that you will have no choice for any of the $\tilde{a}_n$, they will *all* be determined. Observe also that, in the general expression for $\tilde{a}_{2k}$, you will encounter the expression $(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k})$. Don't try to simplify this sum, lest you hide the general pattern.

**15.** This example illustrates the typical reason why you cannot expect the power series method to work for irregular singular points: Take the ODE

$$x^2y'' - y' + y = 0$$

and try $y = \sum_n a_n x^{n+s}$ with $a_{-1} = a_{-2} = \ldots = 0$ and $a_0 \neq 0$. What is the recursion relation you get for the $a_n$? Be sure to fill in the table in # 10 for this example. Determine $s$ from $a_{-1} = 0$, $a_0 \neq 0$.

To check the convergence of the power series, calculate $\lim_{n\to\infty} |\frac{a_{n+1}x^{n+1}}{a_n x^n}|$ and use the ratio test (see manuscript). Does the series converge for any $x \neq 0$?

To understand how solutions may look at an irregular singular point, try the 1st order ODE $x^2y' - ay = 0$, where you can find solutions by means of separation of variables (remember M231?) Sketch a graph of a typical solution in a neighbourhood of 0.

**16.** Let $f(x, y, t) := u(r)\cos\omega t$, where $r = \sqrt{x^2 + y^2}$. Physics tells us the following: For $f(x, y, t)$ to describe a vibrating membrane, with $f(x, y, t)$ telling how high the membrane is above the rest position at point $(x, y)$ and at time $t$, $f$ must satisfy the condition

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = c^{-2}\frac{\partial^2 f}{\partial t^2}$$

with $c$ the wave propagation speed in the membrane.

Your job is simply to use the multivariable chain rule and obtain an ODE for $u$ from the partial differential equation for $f$. Compare the ODE so obtained with the Bessel ODE.