The cases $a = \pm 1$, $b = 0$ lead to $c = \pm 1, d = 0$ and give the units ± 1 in the ring $\mathbb{Z}[\sqrt{-5}]$

Let's pursue the remaining case $a = 0$: Then either $b = 0$ or else:

$c = 0$, $d = -\frac{1}{5b}$ which is impossible for integers b

But $a = b = 0$ as well is impossible for

$$(a + b\sqrt{-5})(c + d\sqrt{-5}) = 1$$

Hence the only units are ± 1.

- Now we show that 3 is prime in the ring $\mathbb{Z}[\sqrt{-5}]$

assume $(a + b\sqrt{-5})(c + d\sqrt{-5}) = 3$

Then, multiplying with the complex conjugate,

$$(a - b\sqrt{-5})(c - d\sqrt{-5}) = 3,$$

we get $(a^2 + 5b^2)(c^2 + 5d^2) = 9.$

This means $a^2 + 5b^2$ can only be among $\{1, 3, 9\}$

There are three ways this can happen:

$$\begin{aligned} a &= \pm 1, \quad b = 0 && (a^2 + 5b^2 = 1) \\ a &= \pm 3, \quad b = 0 && \multirow{2}{*}{} \\ a &= \pm 2, \quad b = \pm 1 && \end{aligned} \Big\} \ (a^2 + 5b^2 = 9)$$

The same reasoning applies to c, d.

But this means, both $a^2 + 5b^2$ and $c^2 + 5d^2$ are among $\{1, 9\}$. With their product being 9, one of them has to be 1. So either $a + b\sqrt{-5} = \pm 1$, or $c + d\sqrt{-5} = \pm 1$

Hence 3 is a prime in $\mathbb{Z}[-\sqrt{5}]$.