The key issue of the following sample solution is in the logic, i.e., in the text provided with the calculation. The left column gives the solution, the right column adds further comments to clarify questions you may have.

Problem: Find all those real numbers x for which

$$(*) \qquad \sqrt{x} \leq 6 - x$$

Solution:

(1)	First note that we must have $x \geq 0$, b/c else $\sqrt{x}$ would not exist as a real number.	and allowing complex numbers wouldn't help, b/c there is no notion of $\leq$ or $\geq$ among complex numbers that are not real.
(2)	By adding $x-6$ to both sides, $(*)$ is equivalent to	'equivalent' means that one inequality is true if and only if the other is; in other words, it is precisely the same x that satisfy either inequality.
(3)	$x + \sqrt{x} - 6 \leq 0$	
(4)	By factoring the left hand side, we get the equivalent form	Note that $\sqrt{x} \cdot \sqrt{x} = x$, but that $\sqrt{x \cdot x}$ is only equal to x if we assume $x \geq 0$. Here we have used the first equation, which had implicitly assumed $x \geq 0$, see (1), so the subtlety pointed out here is not really an issue in our argument.
(5)	$(\sqrt{x} - 2)(\sqrt{x} + 3) \leq 0$.	
(6)	It is permissible to divide inequalities by a *positive* quantity, and $\sqrt{x} + 3$ is positive, because $\sqrt{x}$ is defined to be nonnegative.	When we (informally) say 'is permissible', we mean that the inequality resulting from this division is equivalent to the original inequality. — This step saves some work: if we hadn't noted beforehand that $\sqrt{x} + 3$ is always positive, we would have begun to distinguish various cases from (5), depending on the sign of either factor.
(7)	We obtain $\sqrt{x} - 2 \leq 0$, still equivalent to $(*)$.	
(8)	Equivalently, $\sqrt{x} \leq 2$.	
(9)	Both sides of the inequality are non-negative. Squaring this inequality, we find that (8) *implies* $x \leq 4$.	Two important things happen here. We say 'implies' to denote a conclusion that may not be reversible. We argue that *if* (8) is true, then so is (9), but we do not claim the converse. (9) may be true without (8) being true. For instance, all negative numbers satisfy (9), but they don't satisfy (8). In other words, in this step we are importing 'counterfeit' solutions. — Moreover, it is *not* automatically true that $a \leq b$ implies $a^2 \leq b^2$. For instance $-2 \leq 1$, but $(-2)^2 \not\leq 1^2$. Squaring an inequality does give a correct implication when both sides are non-negative, and this is the case in our situation, and we have used this fact in the argument.
(10)	Together with (1) it follows $0 \leq x \leq 4$.	At this moment, we have reasoned that only these x have a chance of solving $(*)$: in other words, *if* x solves $(*)$, then (10) holds. While it *is* true that these $x \in [0, 4]$ do solve $(*)$, we have not yet made this case. The step from (8) to (9) might have introduced counterfeit solutions, and we have yet to argue why this did not happen in this specific case.

(11)	We may infer an inequality by taking the square root, if all memebers of the inequality are non-negative. Therefore $0 \le x \le 4$ implies $0 \le \sqrt{x} \le 2$.	In this step, we are now *assuming* that $0 \le x \le 4$, and we want to conclude that these x do indeed satisfy $(*)$, in other words, we want to argue that none of the x's in (10) are counterfeit solutions.
(12)	In particular $\sqrt{x} \le 2$, which is (8).	We are dropping the information $0 \le \sqrt{x}$ here, because we are not going to use it any more. (It is true for any $x \ge 0$, regardless of whether x solves a certain inequality like $(*)$ or not.)
(13)	Since (8) has been shown to be equaivalent to $(*)$, we have now concluded that the $x \in [0,4]$ indeed solve $(*)$.	From (1) through (10) we concluded that *only* $x \in [0,4]$ have a chance to solve $(*)$. From (11) on, we have concluded that those x indeed do solve $(*)$. By carefully noting in as many steps as possible that the conclusion goes in both directions, i.e., that statements are equivalent, we could save the labor of going each step back from (8) to $(*)$ backwards separately. That labor was done in the first part of the argument already and needn't be repeated.

Homework:

(1) Now provide a variant of this proof (left column suffices), which attempts to square $(*)$ right away, putting the appropriate text around it all.

(2) Solve the problem $x - \sqrt{x} \le 6$ in a similar way.

(3) Try to solve $(*)$ $\frac{-2}{x+3} \le x$ with careful explanations of what follows from what. As was hinted at during the class meeting, you may need to distinguish two cases: One part of the argument will deal exclusively with those x for which $x + 3 > 0$ and which satisfy $(*)$, another part should exhibit those x satisfying $(*)$ for which $x + 3 < 0$.