Homework Chapter 1
UTK – M251 – Matrix Algebra
Spring 2019, Jochen Denzler

1. Which of the following are linear equations in the unknowns x_1, x_2, x_3, or x, y, z, as applicable:

(a)	$x_1 + x_2^2 + 3x_3 = 7$	(b)	$\frac{x}{2} + 2y - 3z = 0$
(c)	$\sqrt{2}x_1 - 3x_2 + x_3 = 5$	(d)	$2\sqrt{x_1} - x_2 + 8x_3 = 2$
(e)	$x + yz = 2$	(f)	$\frac{2}{x} + 2y - 3z = 0$

2. Find the augmented matrix for the following system of linear equations:

$$\begin{aligned} &5x + 3y - 2z + 11w = 9 \\ &3x - 4w = 11 \\ &x + y + z = \sqrt{5} \end{aligned}$$

3. For which values of the constant k does the system

$$\begin{aligned} x + 2y &= 7 \\ 3x + 6y &= k \end{aligned}$$

have (a) no solution (b) exactly one solution (c) infinitely many solutions?

4. Solve each of the following systems by Gauss–Jordan elimination:

$$(a)\ \begin{aligned} 3x + 2y - 2z &= 1 \\ -x + y + 2z &= 2 \\ 2x - 3y \qquad &= 1 \end{aligned} \qquad (b)\ \begin{aligned} 3x_1 - 2x_2 - x_3 - x_4 &= 0 \\ 2x_1 + x_2 \qquad + 3x_4 &= 1 \end{aligned} \qquad (c)\ \begin{aligned} 2x + y + 3z - w &= 4 \\ x - 3y + 2z + 4w &= 1 \\ 4x + 9y + 5z - 11w &= 7 \end{aligned}$$

5. Solve (a) in the previous problem by Gauss elimination and back substitution.

6. For which values of λ does the system

$$\begin{aligned} (3 - \lambda)x + y &= 0 \\ 3x + (5 - \lambda)y &= 0 \end{aligned}$$

have nontrivial solutions? (Note: Later in the semester, this type of problem will become known as an "eigenvalue problem" and will be studied in more generality.)

7. Give an example of a linear system of equations that has more equations than unknowns, but still has infinitely many solutions.

8. Give an example of a linear system of equations that has more unknowns than equations, but has no solutions.

9. Find numbers a, b and c such that the curve $y = ax^2 + bx + c$ passes through the points $(1, 3)$, $(2, 3)$ and $(4, 9)$. (Note that this problem leads to *linear* equations even though $f(x) = ax^2 + bx + c$ is a *non*linear function of x.)

10. obsolete

11. Suppose the following matrices have the sizes given:

matrix	A	B	C	D	E
size	3×7	7×2	3×7	3×2	2×3

Determine the sizes of the following matrices, or say "undefined", if this is the case. AB, BA, $A - C$, $A^T B$, $A^T C$, $D^T A$, $CB + D$, DE, $DD^T + E^T E$, $(A + C)BE$

12. Compute

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & 2 & 1 & -3 \\ 3 & -1 & 7 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ -1 & 3 \\ 7 & -4 \end{bmatrix}$$

13. Compute AB and BA for

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -4 \\ 3 & -1 \end{bmatrix}$$

14. Compute A^2 (a shorthand for AA) if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

15. In the following matrix expression, $*$ denotes numbers that I decline to reveal to you. Compute as many entries of the result as is possible, based on this limited information, and fill in $*$ for the entries that cannot be determined.

$$\begin{bmatrix} * & * & * & 0 & * \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & * & 0 \\ -1 & * & 0 \\ 2 & * & 0 \\ 4 & 0 & 0 \\ -1 & * & 0 \end{bmatrix}$$

16. Write the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 7 & 7 & 7 \\ -\frac{1}{5} & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 88 \\ 888 \end{bmatrix}$$

as a system of linear equations.

Also write the system of linear equations whose augmented matrix can be found on page 1 of the introductory notes as a matrix equation; you may invent your favorite names for the unknowns.

17. For the matrices

$$A = \begin{bmatrix} -2 & 1 \\ 0 & \frac{3}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix},$$

calculate the following expressions. If some expression is not defined, say so and explain why it is not defined.

$$(a)\ (A + B^T)C\,, \qquad (b)\ A^T A + C\,, \qquad (c)\ B + CA\,, \qquad (d)\ A^T - 2CB$$

18. For the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ calculate $\operatorname{tr} A^T A$ and $\operatorname{tr} AA^T$. (If you are efficient, you need to calculate only part of $A^T A$ and of AA^T.)

19. Find a square matrix A such that $\operatorname{tr}(A^2) \neq (\operatorname{tr} A)^2$. (Unless it's your really unlucky day, your first best random choice of a 2×2 matrix should do the trick.)

20. Suppose that a square matrix A satisfies $A^2 - 3A + 2I = 0$. Show that A is invertible, and that $A^{-1} = \frac{3}{2}I - \frac{1}{2}A$.

21. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

Calculate $(A + B)^2$ and $A^2 + 2AB + B^2$, and explain how it comes that they are not the same. – Give the *correct* binomial formula for matrices:

$$(A + B)^2 = A^2 + B^2 + \underline{\hspace{3cm}}\ ?$$

22. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

23. Find the inverses of the following matrices:

$$B = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & & & & \\ & \frac{4}{5} & & & \\ & & 2 & & \\ & & & \pi & \\ & & & & -3 \end{bmatrix}$$

(Note: Vacant entries are understood to be 0. This convention is popular when dealing with large matrices that have only a comparatively small number of non-zero entries, so-called sparse matrices.)

24. Find an (the) elementary matrix E for which $A = EB$ when

$$A = \begin{bmatrix} 8 & 3 & 3 \\ 4 & 2 & 1 \\ -5 & 2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 3 & 3 \\ 4 & 2 & 1 \\ 3 & 6 & -1 \end{bmatrix}$$

25. Solve the linear system

$$\begin{aligned} -x - 2y - 3z &= 2 \\ w + x + 4y + 4z &= 5 \\ w + 3x + 7y + 9z &= -1 \\ -w - 2x - 4y - 6z &= 3 \end{aligned}$$

by actually calculating the inverse of the coefficient matrix and then evaluationg the matrix product. (Remember from the lecture that this method is usually not encouraged, and not efficient for a single system of linear equations. Just once so that you see it work in practice.)

26. Which of the following matrices are invertible? (Decide by eyeballing; you are not required to find the inverse)

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 9 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 7 & 5 \end{bmatrix}$$

27. I have done the following calculation for you already. You job is to determine, by eyeballing alone, if the two matrices on the left hand side commute or not; and to explain your reasoning in one or two sentences. If you find it difficult to come up with a compelling argument, you may need to study the glossary (again).

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 11 & 5 & -2 \\ 5 & 0 & 15 \\ -2 & 15 & -11 \end{bmatrix} = \begin{bmatrix} 15 & 50 & -5 \\ 15 & 25 & -30 \\ 38 & 15 & 9 \end{bmatrix}$$

28. In the following calculations, the $*$ represent entries which I do not reveal to you. Carry out the evaluations to the extent possible and put $*$ for entries that cannot be determined with the given information. Note that vacant entries are understood to be zero.

$$\begin{bmatrix} 1 & * & * \\ & 2 & * \\ & & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & * & * \\ & 5 & * \\ & & 4 \end{bmatrix} = ?, \qquad \begin{bmatrix} 1 & * & * \\ & 2 & * \\ & & 3 \end{bmatrix}^{-1} = ?$$

29. Find a diagonal matrix A that satisfies

$$A^{-4} = \begin{bmatrix} 16 & & \\ & 1 & \\ & & 1/9 \end{bmatrix}$$

(I didn't mention this in class, but hope you find the notation obvious:
$A^{-4} := A^{-1} \cdot A^{-1} \cdot A^{-1} \cdot A^{-1}$.)

30. A and B are given square matrices of the same size (they may or may not be symmetric). Which of the following expressions will always produce a symmetric matrix, whatever matrices A and B I may have in mind?

$$(a) \quad AB + BA, \qquad (b) \quad A^T A, \qquad (c) \quad AB - BA, \qquad (d) \quad ABA^T B^T$$

31. Answer the same question as before, but this time with A and B given *symmetric* matrices of the same size.

32. Find lower and upper triangular matrices L and U such that $A = LU$, where A is the matrix

$$A = \begin{bmatrix} 2 & 8 & -2 \\ -1 & -1 & -14 \\ 2 & 11 & -19 \end{bmatrix}$$