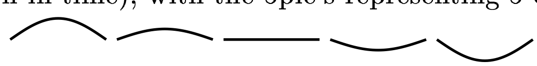
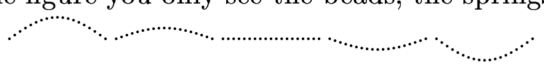
Homework Chapter 5
UTK – M251 – Matrix Algebra
Spring 2006, Jochen Denzler, MWF 1:25–2:15, Ayres 102

0. (numbered 0 for backward compatibility with earlier course) *This problem wants to help you better understand why certain sets of functions deserve to be called (treated as a) vector space, with the effect that the functions therein are then called 'vectors'. To this end, it shows you how the classical vectors $\boldsymbol{x} \in \mathbb{R}^n$ can be viewed as functions $s \mapsto \boldsymbol{x}_s$ defined only for $s = 1, 2, \ldots, n$. And as an extra benefit, it does so in the context of a genuine application of eigenvalues. — And reading this thing is a significant part of your hwk.*

Think of a guitar string. As it vibrates, the frequency will determine the pitch of the sound. We will only study the shape of the the string. Typically it will look like this (going up and down in time), with the 5pic's representing 5 different times:



We do not care about the vibration itself, but about the shape, which is a sine curve, as you can guess from the picture. Whether it's the graph of $y = 2\sin s$ or $y = \sin s$ or $y = -\sin s$ etc depends on the time, and we don't care about the constant. The math to really do this problem is in Math435, so we cannot do it here. A similar, but simpler problem is to replace the guitar string with a collection of beads, connected by thin springs. In the figure you only see the beads, the springs are too fine to be seen:



Now the physics bigshots will tell you the following, with all due simplifications: The shape of the string of beads is given by an eigenvector $\boldsymbol{y}$ of the $n \times n$ matrix

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

where the s^{th} entry y_s of $\boldsymbol{y}$ tells how high the s^{th} bead is up. So you see: you can graph a vector in $\mathbb{R}^n$ by plotting n dots, like a function! The physics bigshots will also tell you that the square root of the eigenvalue of A is the frequency with which the string of beads oscillates up and down.

Now here is your job: I want you to *illustrate* what the physics bigshots are telling here.

(a) For A as a 7×7 matrix, I want you to check that $4\sin^2 \frac{\pi}{16}$ is an eigenvalue with eigenvector

$$\boldsymbol{y} = [\sin\frac{\pi}{8}, \sin\frac{2\pi}{8}, \sin\frac{3\pi}{8}, \sin\frac{4\pi}{8}, \sin\frac{5\pi}{8}, \sin\frac{6\pi}{8}, \sin\frac{7\pi}{8}]^T$$

and I want you to plot this string of beads. You may wish to use technology for this.

(b) For the same A, I want you to check that $4\sin^2 \frac{2\pi}{16}$ is also an eigenvalue, this time with eigenvector

$$\boldsymbol{z} = [\sin\frac{2\pi}{8}, \sin\frac{4\pi}{8}, \sin\frac{6\pi}{8}, \sin\frac{8\pi}{8}, \sin\frac{10\pi}{8}, \sin\frac{12\pi}{8}, \sin\frac{14\pi}{8}]^T$$

and I want you to plot this string of beads as well.

You see, the string of beads is capable of oscillating in more than one different shape (and I could have made more parts, with yet more shapes). For the guitar string the same is also true, and in music they call this phenomenon overtones, because to the different shapes there belong different frequencies ($\sqrt{\text{eigenvalue}}$), i.e., different pitches.

1. Check if the following set of vectors in $\mathbb{R}^4$ is linearly independent. Show your work. $S = \{[0,\ 1,\ 2,\ 3]^T,\ [1,\ 2,\ 3,\ 4]^T,\ [0,\ 2,\ 0,\ 2]^T\}$

2. Explain in one sentence why it can be seen at a glance that the set in the previous example is not a basis of $\mathbb{R}^4$.

3. Is $S = \{[1,\ 1,\ 1]^T,\ [2,\ -2,\ 1]^T,\ [4,\ 0,\ 3]^T\}$
(a) linearly independent?
(b) a spanning set for $\mathbb{R}^3$?
(c) a basis for $\mathbb{R}^3$?

Each answer requires an explanation.

4. Show that $S = \{[1,\ 1,\ 1]^T,\ [2,\ -2,\ 1]^T,\ [1,\ 0,\ 1]^T\}$ is a basis of $\mathbb{R}^3$.

5. Show that the set $S = \{x^2+1, 2x, 2x^2-3\}$ is a basis of the vector space P_2 of polynomials of degree at most 2.

6. Is the set $S = \{1, \sin x, \cos x\}$ linearly independent in the vector space $C^0(\mathbb{R})$ of continuous real functions? – Hint: You have to address the question: If $k_1 1 + k_2 \sin x + k_3 \cos x = \mathbf{0}$ (this $\mathbf{0}$ means the zero *function*), in other words, if $k_1 1 + k_2 \sin x + k_3 \cos x = 0$ for *every* x, can we conclude that $k_1 = k_2 = k_3 = 0$?

7. Is the set $S = \{1, \sin^2 x, \cos^2 x\}$ linearly independent in the vector space $C^0(\mathbb{R})$ of continuous real functions?

8. Find the coordinate vector of the vector $[1,\ 2,\ 3]^T$ with respect to the basis $S = \{[1,\ 1,\ 1]^T,\ [2,\ -2,\ 1]^T,\ [1,\ 0,\ 1]^T\}$ of $\mathbb{R}^3$ from pblm. 4.

9. Find the coordinate vector of the polynomial $x^2 + x + 1$ with respect to the basis $S = \{x^2 + 1, 2x, 2x^2 - 3\}$ from pblm. 5.

10. Show that if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are non-zero vectors in $\mathbb{R}^3$ that are mutually orthogonal, then $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is linearly independent.

11. According to Thm. 5.2.2 in the book, the set of solutions to a *homogeneous* linear system in n unknowns is a linear subspace of $\mathbb{R}^n$, called the solution space of the linear system.

Find a basis for the solution space of the homogeneous linear system, and determine the dimension of the solution space.

$$\begin{aligned} 3x_1 - x_2 - x_3 + 3x_4 &= 0 \\ 4x_1 + x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

12. Same question for the system

$$\begin{aligned} 3x_1 - x_2 - x_3 + 3x_4 &= 0 \\ 4x_1 + x_2 - x_3 + 2x_4 &= 0 \\ x_1 + x_2 + x_3 - 4x_4 &= 0 \end{aligned}$$

13. The real vector space $C^2(\mathbb{R})$ contains all twice differentiable real functions whose second derviative is still continuous. In M231 you have learned (or will learn) that a function $y = f(x)$ satisfies the condition $f''(x) - 3f'(x) + 2f(x) = 0$ for all x, if and only if f is of the form $f(x) = c_1 e^x + c_2 e^{2x}$. You may just take this fact for granted in this class, and now this should be an *easy* problem:

Give a basis for the vector space W consisting of those functions that satisfy $f''(x) - 3f'(x) + 2f(x) = 0$ for all x; and determine the dimension of W.

14. What is the dimension of $\mathbb{R}^{3\times 3}$ (also known under the name M_{33}, the vector space of all 3×3 matrices? Give two different bases for this vector space.

15. $S_0^{3\times 3}$ is the set of all *symmetric* 3×3 matrices whose trace is 0. Make sure that you understand that $S_0^{3\times 3}$ is a subspace of $\mathbb{R}^{3\times 3}$. What is the dimension of $S_0^{3\times 3}$? Give a basis. (If you have forgotten the definition of the trace, it's early in the glossary, or use the index in the book.)

16. Draw into one single picture of the plane $\mathbb{R}^2$ the following objects:

(a) The solution set of the *homogeneous* linear system $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (with each solution being represented as a *point* in $\mathbb{R}^2$).

(b) The solution set of the *inhomogeneous* linear system $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$ (with each solution being represented as a *point* in $\mathbb{R}^2$).

(c) For two particular solutions of the inhomogenous system from part (b), according to your free choice, highlight the point representing them in color, and also draw the respective *vectors* from the origin to these points in the same color.

I want a decent picture that you would not be ashamed to xerox for your class if you were the teacher.

17. Study sections 5.5 and 5.6 as needed; let

$$A = \begin{bmatrix} 3 & 2 & 1 & -1 & 4 \\ 5 & 8 & 7 & 3 & 6 \\ -2 & 1 & 2 & 3 & -3 \end{bmatrix}$$

Invent one non-zero vector that is in the row space of A, but is not among the rows of A. Also determine the dimension of the row space of A by giving an explicit basis for it.

18. Same matrix A as before. Determine a basis for the column space of A. (There are many correct solutions of course.) Give the dimension of this column space.

19. Same matrix A as before. Is the vector $[1,\ 1,\ 1]^T$ in the column space of the matrix A? Show your work. (No credit for merely saying yes/no.)

20. Same matrix A as before. Determine the null space of A.

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21. Draw the vectors $\boldsymbol{u} = [2,\ 1]^T$ and $\boldsymbol{v} = [-1,\ 1]^T$ in the plane. (Again we need a decent figure that you would not be ashamed to xerox for your class if you were the teacher. A unit length of 1 or 1.5 inches is perfectly appropriate.) First rotate each of them by 60 degrees counterclockwise (geometrically, using ruler and protractor), then reflect the resulting vectors in the line $x_2 = -2x_1$. You should have one line and six vectors in your figure now. In a different color, I want you to start over with $\boldsymbol{u}$ and $\boldsymbol{v}$, but this time first reflect them in the line $x_2 = -2x_1$, and then rotate the result counterclockwise by 60 degrees. That will add 4 more vectors in a different color to the figure.

22. Same $\boldsymbol{u}$ and $\boldsymbol{v}$ as before. Refer back to the figure from the previous problem.

Find the matrix Q that represents the rotation by 60 degrees counterclockwise, and the matrix S that represents the reflection in the line $x_2 = -2x_1$. (You had one problem of this type in Chapter 4 already and may refer back for help.) Calculate both products QS and SQ. Label all vectors in the previous problem with $\boldsymbol{u}$, $\boldsymbol{v}$, $Q\boldsymbol{u}$, $S\boldsymbol{v}$, $QS\boldsymbol{u}$, etc., as appropriate. Calculate the vectors $QS\boldsymbol{u}$ and $SQ\boldsymbol{u}$ algebraically and compare with the figure.

23. Calculate the determinant

$$\begin{vmatrix} a & 2 & 3 & b \\ 1 & -1 & 2 & 5 \\ 2 & 2 & -1 & c \\ d & 2 & 3 & -2 \end{vmatrix}$$

in a reasonably efficient way using at least one row or column operation and at least one cofactor expansion of some row or column.