

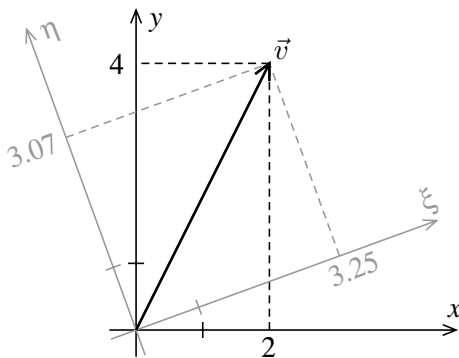
Notes on Vector Calculus for M247

Remark: These notes are organized differently than the lecture; the lecture gave you a panorama of examples and motivations, with little attempt to put things in a systematic order. Here I strive to group by similar themes, to put the pieces together into a bigger picture.

1 Vector operations

1.1 Preliminary remarks

First I should note that, whenever we write vectors in terms of components, these components refer to a *cartesian* coordinate system: coordinate axes are mutually orthogonal and have the same unit distance for their tick marks. You can still rotate the coordinate axes, and then the vector will have different components in the rotated coordinate system, but it would still refer geometrically to the same object. When we talk about vector operations, we talk about operations like vector addition, dot product, or cross product, which make geometric sense without reference to a specific cartesian coordinate system.



In the xy coordinate system, the vector \vec{v} has components $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, whereas in the $\xi\eta$ coordinate system, the same vector has coordinates $\begin{bmatrix} 3.25 \\ 3.07 \end{bmatrix}$. (1)

This observation is only being made here to stress the geometric nature of vectors; since practical calculations will be made in one specific coordinate system, you will not have to worry about questions like ‘what are the correct components of this vector?’

The word ‘scalar’ is a synonym for ‘number’ that is used in a context of vectors. So “scalar” means “number (not vector)”.

1.2 Operations, geometrically and algebraically

LINEAR OPERATIONS: The easiest operations are addition of vectors and multiplication of a vector by a scalar. You know them, and I need not repeat them here.

NORM: The norm of a vector is a scalar that geometrically denotes the length of that vector. In coordinates, it is calculated by means of Pythagoras’ theorem. For instance, the vector \vec{v} in (1) has norm $\|\vec{v}\| = \sqrt{2^2 + 4^2} = \sqrt{3.25^2 + 3.07^2} = 4.47$. This example goes to show that the norm of a vector, as defined by the Pythagoras formula, is geometrically meaningful because it does not depend on which coordinate system we choose even though the components do depend on the chosen coordinate system.

DOT PRODUCT: Next is the dot product (in \mathbb{R}^2 , or \mathbb{R}^3 , or any \mathbb{R}^n for that matter). Geometrically it is defined as $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \varphi$, where φ denotes the angle between the two

vectors. Algebraically, it can be calculated as $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1v_1 + u_2v_2$ (for vectors in \mathbb{R}^2). Similarly with more components in \mathbb{R}^3 . Earlier in class, we had used the law of cosines to show that the geometric and the algebraic definition coincide.

The main application of the dot product in physics is *work*: It is $\vec{F} \cdot \vec{s}$ where \vec{F} is the force vector, and \vec{s} is the displacement which we cause by means of this force. Their dot product is the work that we need to do in order to achieve this displacement.

CROSS PRODUCT: The cross product applies only to vectors in \mathbb{R}^3 , not other \mathbb{R}^n . The cross product of two vectors is a vector again, and it has the specific property that order of factors does matter: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. (We skip other interesting properties of the cross product here. See Hwk 44-46, and you may refer to the posted Linear Algebra glossary pg 11).

When dealing with the cross product, we should use a *right handed* cartesian coordinate system: i.e., if the thumb and forefinger of your right hand point in the direction of the 1st and 2nd coordinate axes respectively, then the middle finger should point in the direction of the 3rd coordinate axis.

In a right handed cartesian coordinate system, the cross product is defined as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

The cross product $\vec{u} \times \vec{v}$ is geometrically characterized by the following properties: Its norm is the area of the parallelogram spanned by \vec{u} and \vec{v} (that would be 0, if these two vectors point in the same or opposite directions). Its direction is orthogonal to the plane spanned by the two vectors. (If the two vectors point in the same or opposite directions, they don't span a plane, but then we also do not need a direction because the norm of $\vec{u} \times \vec{v}$ is 0.) These two properties determine two vectors, one of which is $\vec{u} \times \vec{v}$, and the other is $\vec{v} \times \vec{u}$. If you use your right hand in hitchhiker's position, with 4 fingers pointing in the direction in which you have to turn \vec{u} to bring it towards \vec{v} , then the thumb tells you which of the two vectors selected by the geometric properties is $\vec{u} \times \vec{v}$.

The main application for our purposes lies in the fact that $\|\vec{u} \times \vec{v}\|$ gives the area of the parallelogram spanned by \vec{u} and \vec{v} . In physics, the Coriolis force in rotating systems, and the Lorentz force on charged particles moving through a magnetic field, are examples that use the cross product, but we need not pursue these examples here.

CROSS PRODUCT AND 2 DIMENSIONS: While, strictly speaking, there is no cross product in \mathbb{R}^2 , we can use the cross product on vectors in \mathbb{R}^2 by adding a third component 0, as if we were viewing the xy plane as part of \mathbb{R}^3 . So we turn the vector $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ into a vector

$\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ and calculate

$$\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

This means that $u_1v_2 - u_2v_1$ is plus or minus the area of the parallelogram spanned by $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Or in other words, the area is the absolute value $|u_1v_2 - u_2v_1|$. We call $u_1v_2 - u_2v_1$ (without the absolute value) the *oriented area* of the parallelogram in question.

SCALAR TRIPLE PRODUCT: We have discussed in class that the scalar triple product $(\vec{u} \times \vec{v}) \cdot \vec{w}$ gives plus or minus the volume of the parallelepiped ('tilted box') whose sides are \vec{u} , \vec{v} and \vec{w} . In components, one calculates

$$\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = u_1 v_2 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_1 v_3 w_2 - u_2 v_1 w_3 - u_3 v_2 w_1$$

This formula implies for instance $(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u}$ and many formulas of a similar nature.

So, concludingly, the volume of a parallelepiped is the absolute value of the scalar triple product of its sides; the scalar triple product without the absolute value is called *oriented volume* of the parallelepiped.

DETERMINANTS: In Linear Algebra, a certain number is assigned to each square matrix, and it is called its determinant. The determinant of a 1×1 matrix $[a]$ is simply its sole entry a . The only reason we want this case is for later comparison of multi-variable results with single-variable results.

The determinant of a 2×2 matrix is defined as $\det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = u_1 v_2 - u_2 v_1$, which is just the formula for the oriented area encountered above.

The determinant of a 3×3 matrix is defined as the scalar triple product of its columns, and its geometric meaning is an oriented volume as outlined above.

Determinants of larger matrices will be ignored to us; their definition and practical calculation is studied in Linear Algebra. They do describe higher dimensional analogs of volume and occur in an obvious way when generalizing to more variables..

2 Vector Fields and Scalar Fields

Physicists like to call multivariable functions whose values are just numbers by the name 'scalar fields'. Like in a corn field, which has a stalk of corn at (kind of) every place, a temperature field describes a temperature at every point in a certain domain. So 'scalar field' is merely a fancy name for a real-valued function defined in some domain of \mathbb{R}^2 or \mathbb{R}^3 .

In contradistinction, vector fields are vector valued functions in a domain of \mathbb{R}^3 or \mathbb{R}^2 . To each point in some domain of \mathbb{R}^2 a 2-component vector is assigned, or to each point in some domain of \mathbb{R}^3 , a 3-component vector is assigned. Typical examples in applications are the following:

A velocity field may describe the velocity of a fluid (a vector) at every point in the domain in which this fluid is flowing. The gravitational field describes the force (per unit mass) that an object experiences due to gravitation. Similarly, an electric field describes the electrostatic force (per unit charge) that a certain charge distribution may cause.

Practically we would restrict the term 'vector field' to a situation where these multi-component (mathematical) vectors have a geometric meaning. For instance the function from \mathbb{R}^3 to \mathbb{R}^3

named 'weather', which assigns to each point (x, y, z) the 3×1 matrix $\begin{bmatrix} \text{temperature}(x, y, z) \\ \text{pressure}(x, y, z) \\ \text{humidity}(x, y, z) \end{bmatrix}$

is a perfectly good function to study, but we wouldn't want to call it a vector field, because

the components do not have a geometric meaning as components of some unified entity in a cartesian coordinate system. This restriction is of a pragmatic nature and need not bother you from the point of view of mathematical theory. It only means that the vector calculus you are to learn here will not be useful when studying the above ‘weather’ function.

3 Integrals over domains in the plane and in 3-space

3.1 Definitions, and Fubini

For theoretical purposes, we consider functions in \mathbb{R}^2 that are bounded and are continuous except on a set that is the finite union of graphs of continuous single variable functions. It can be shown that for such functions, a Riemann integral over rectangles exists. This Riemann integral is defined as a limit of Riemann sums associated with partitions of the rectangle over which we are integrating. Apart from this general outline, we have not focused on further theory. The theoretical hypothesis about continuity except on a certain set, which is made here, is awkward, and less general than desired, but it works for our needs. A decent theoretical foundation is best left to more advanced courses.

We write such an integral as $\int_{[a,b] \times [c,d]} f(x,y) d(x,y)$ and call $d(x,y)$ the area element. This symbol represents the small pieces of area in the Riemann sums, namely rectangle areas $\Delta x \Delta y$. Notations for the area element vary widely in the literature: you’ll find $dx dy$ or d^2x . I will often simply write $darea$ to stress the geometric idea apart from a specific notation. $\int_{[a,b] \times [c,d]} f(x,y) d(x,y)$ describes the volume beneath the graph of f and above the rectangle $[a,b] \times [c,d]$.

Practical evaluation of such integrals can be done by repeated single-variable integrals. The Fubini theorem states (under the hypotheses made on f) that the order of integration is arbitrary:

$$\int_{[a,b] \times [c,d]} f(x,y) d(x,y) = \int_a^b \left(\int_c^d f(x,y) dy \right) dx = \int_c^d \left(\int_a^b f(x,y) dx \right) dy$$

This means that all practical evaluation of multi-variable integrals is ultimately reduced to repeated single-variable integrals. Our focus will therefore be not so much practical evaluation, but rather the questions of how multi-variable integrals show up in applications and what they mean in practice.

We will not be content with integration domains that are rectangles. Our integration domains can be discs, triangles, or odd-shaped domains in the plane. This is why our theoretical hypothesis insisted to allow some discontinuities in the integrand f . Instead of integrating f over some odd shaped domain U (that is at least contained in some big rectangle $[a,b] \times [c,d]$), we construct the function g , such that $g(x,y) = f(x,y)$ for $(x,y) \in U$ and $g(x,y) = 0$ for $(x,y) \notin U$. Then we define $\int_U f(x,y) d(x,y) := \int_{[a,b] \times [c,d]} g(x,y) d(x,y)$.

It takes a bit of training to choose the correct limits of integration for the single-variable integrals when integrating over non-rectangular domains. Hwk problems 42, 43 illustrate this issue. There is no mechanical way of doing this; it is more like in a word problem, where you need to translate the geometric description of U into formulas. While the integration is permitted in any order, the limits of integration may look different depending on the order chosen. In $\int_*^* (\int_*^* \dots dx) dy$ the limits of the inner integral (over x) will typically depend on y , but the limits of the outer integral over y cannot depend on any of the variables any more. If

you integrate in the opposite order $\int_*^* (\int_*^* \dots dy) dx$, the limits of the x integral (which is the outer integral now) cannot depend on variables any more, whereas the limits for the (inner) y integral may now depend on x .

The same principles apply to integrals over domains in \mathbb{R}^3 , which can be resolved into three repeated single-variable integrals. These can be done in either order, with due adjustment of the limits. The volume element $dvol$ can be written as $d(x, y, z)$ or $d^3\vec{x}$ and, in cartesian coordinates it is written as a product $dx dy dz$.

3.2 Applications

In many good applications, the function f under the integral is fairly simple, functions so simple that we would hardly have bothered with them in single variable calculus. Computational complexity may still come from the area or volume element, if, for the sake of convenient adjustment to the geometry of the domain over which we integrate, we express this volume element in ‘curvy’ coordinates (like polar or spherical coordinates), or from the limits of integration in the inner integrals.

For instance $f(\vec{x}) \equiv 1$: Then $\int_U 1 dvol$ in 3 dimensions gives the volume of U , and $\int_U 1 darea$ gives the area of a domain in \mathbb{R}^2 . — These are a special case of the following: Think of f as a density (mass per volume) of a material that fills the domain of integration U ; here the material may be heterogeneous, so the density may vary from place to place. Then $f(\vec{x}) dvol = f(\vec{x}) dx dy dz$ is the mass element, i.e., the mass of a little piece of material (of volume $dx dy dz$) about location \vec{x} . ‘Adding these all up’, i.e., taking the Riemann integral $\int_U f(\vec{x}) dvol$, gives the mass of the material filling U .

In the physics of rotating rigid bodies, the *moment of inertia* plays a similar role as the mass does in the physics of small objects moving forwards. This moment of inertia is defined by taking for f the distance square of a point from the axis of rotation, times the density ρ . In other words, a mass element $\rho(\vec{x}) dvol$ that is distance s away from the axis of rotation contributes $s^2 \rho(\vec{x}) dvol$ to the moment of inertia, and the total moment of inertia of the body whose material covers the domain U is therefore $\int_U s^2 \rho(\vec{x}) dvol$.

If you think of f as a temperature, then the quotient $\int_U f(\vec{x}) dvol / \int_U 1 dvol$ gives the average temperature; a quantity that may be physically less meaningful than it seems to be at first sight (in particular if the material filling U consists of different materials, with different thermal properties). But this example may still serve to illustrate the integral.

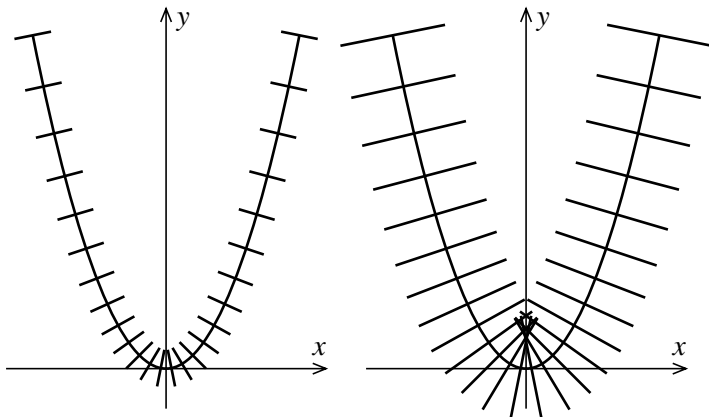
3.3 Change of variables formula

In practice, changing variables in a MV integral often serves a different purpose than changing variable in a SV integral. In MV integrals, the choice of variables often amounts to choosing coordinates that best describe the geometry of a domain U , whereas in SV integrals there is no issue of domain geometry (it’s all intervals), and changes of variables serve to simplify the algebraic structure of the integrand. The latter will still be done in the process of practical evaluation of MV integrals as a sequence of SV integrals. But before getting there, we choose MV coordinates that allow us to describe the limits of integration for the SV integrals conveniently:

If U is a rectangle, or a rectangular box, we choose (2dim or 3dim) cartesian coordinates. If U is a disc, or an annulus (ring), or a sector thereof, we may find it more convenient to choose polar coordinates. For spherical or cylindrical symmetry, we choose spherical or cylindrical

coordinates respectively. This is not a law, but a matter of wisdom and convenience. There are situations, where you would naturally use tailor-made coordinates.

I gave as an example the domain in the plane that is covered if you draw a smooth curve with a thick pen: It consists of those points whose distance to a curve is at most a certain constant. You would most conveniently describe a point by two coordinates: one coordinate u giving a position on the curve, for instance the point (u, u^2) on the parabola in the figure, and one coordinate v giving the distance away from that ‘base point’ on the curve.



This works only if the pen is not too thick; every point in the domain U must be described in terms of a *unique* base point on the curve that is closest to it. In the figure, the coordinates (u, v) are good for the domain on the left, but not for the domain on the right any more, because of the intersection of the ‘different v axes’.

When transforming an integral into different coordinates, you need to re-determine the limits of integration for the new variables and write the area or volume element in terms of the new coordinates. Often you don’t actually ‘transform into’ new coordinates, but simply devise the integral in terms of these coordinates from the onset. The limits of integration are often easy. In the example of the thick pen drawing a parabola, we would just have $-a \leq u \leq a$ (telling how much of the parabola is drawn) and $-b \leq v \leq b$ if the pen has thickness $2b$, with its center following the parabola. (Think how nearly impossible it would be to describe this domain in terms of cartesian coordinates (x, y) .)

The following general formula applies in 2 or 3 dimensions respectively: If $x = x(u, v)$ and $y = y(u, v)$ (or $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$), you calculate the total derivative of this coordinate transformation (often called the Jacobi matrix J):

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

We then have the formulas $dx dy = |\det J| du dv$ or $dx dy dz = |\det J| du dv dw$ respectively. If (x, y, z) were cartesian coordinates in the first place, then $dx dy dz$ is the volume element, and we have expressed the volume element in terms of the curvy coordinates (u, v, w) .

You can try this formula on $x = r \cos \varphi$, $y = r \sin \varphi$ and come up with $darea = dx dy = r dr d\varphi$. This special case (area element in polar coordinates) is what you will probably commit to memory anyways. It is also intuitive, because the small sector of a thin ring is approximated by a rectangle whose one side (along the arc) is $r d\varphi$ and whose other side is dr . It is more the exotic coordinates like in the pen-parabola example, for which you will actually need to use the determinant formula.

In the homework you have calculated the volume element in spherical coordinates based on this transformation formula. You can also obtain the sides of the rectangular box from geometric considerations.

We omit a *formal* proof of the transformation formula. The intuitive reason why it is true

is the following: the area or volume is not taken apart into small rectangles any more, but instead is decomposed into little pieces delimited by curvy coordinate lines; in polar coordinates this would be little sectors of thin annuli (rings). These are approximate rectangles or approximate parallelograms (in 3 dimensions: approximate parallelepipeds), and the *relative* error we make in treating them like exact parallelograms or parallelepipeds will go to 0 as the meshsize goes to 0. So in the limit, which is the Riemann integral, this approximation will be justified. As u changes by a small amount, the point $\vec{x} = \vec{x}(u)$ changes by $(\partial\vec{x}/\partial u) du$. This little change is a vector that makes up one side of the parallelogram. The other side is analogously $(\partial\vec{x}/\partial v) dv$. We use the formula for the area of a parallelogram (or volume of a parallelepiped in 3 dim), and this is how the determinant comes up.

Note that in this calculation we are moving du and dv out of the absolute value as if they were positive quantities:

$$\left| \det \begin{bmatrix} \frac{\partial x}{\partial u} du & \frac{\partial x}{\partial v} dv \\ \frac{\partial y}{\partial u} du & \frac{\partial y}{\partial v} dv \end{bmatrix} \right| = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right| du dv$$

This is correct, because we automatically describe our limits of integration in such a way as to put the smaller number as the lower limit and the larger as the upper limit of integration. So the changes du, dv are positive along the integration.

This convention is at variance with how we are doing SV integrals. There we sometimes allow the upper limit of integration to be smaller than the lower limit. Observe this distinction when comparing the MV substitution formula with the SV substitution formula. For instance take the substitution $x = \cos u$. Then in SV calculus we use $dx/du = -\sin u$ and calculate, for instance

$$\int_{-1}^1 \sqrt{1-x^2} dx = \int_{\pi}^0 \sqrt{1-\cos^2 u} (-\sin u) du = \dots$$

The Jacobi matrix would simply be the 1×1 matrix $J = \left[\frac{dx}{du}\right] = [-\sin u]$. Its determinant equals its one and only entry, $-\sin u$. But for $u \in [0, \pi]$, $\sin u$ is positive, and the absolute value is $|\det J| = \sin u$. The same calculation, written multi-variable style, would therefore look as follows:

$$\int_{[-1,1]} \sqrt{1-x^2} dx = \int_{[0,\pi]} \sqrt{1-\cos^2 u} |-\sin u| du = \dots$$

So the determinant of J really generalizes the dx/du in the substitution of SV integrals, and the absolute value is just an expression of the fact that in MV calculus *we* take care of the orientation, whereas in SV calculus, the integral formalism takes care of the orientation automatically.

(If at some time, you learn MV integration in the language of differential forms, which is BTW what the book by Hubbard & Hubbard is doing, then you have a MV formalism of integration that takes care of the orientation automatically and hence does not have the absolute value around the determinant. However, this approach requires a much heavier load on abstract linear algebra and camouflages the geometric intuition from the sight of individuals learning the material the first time. This is why I use the traditional approach without differential forms in this lecture, as does the vast majority of textbooks.)

4 Integrals over curves and surfaces

4.1 Introduction

If you have tiles, you can tile a (flat) floor or the (curved) belly of the space shuttle. Analogously, once you know the area element, you can integrate functions over surfaces, be they flat (domains in \mathbb{R}^2), or curved (like the surface of a sphere, or part of it). The key issue is that you put a coordinate system on the surface and express the area element in terms of it. Theoretically, there is a vast infinity of choices for such a coordinate system. In practice, there is still a variety of reasonable choices to consider. The value of the integrals over surfaces that we are going to consider will not depend on which coordinates you choose, even though the route and ease of the practical calculational steps towards that result may very well depend on the choice of coordinate system: you can calculate with wise or with unwise choices of coordinates.

The fact that the value of the integrals will not depend on how we parametrize the surfaces (i.e., on which coordinates we use) can be proved as a reasonably straightforward consequence of the change of variables formula. But we will not pursue this proof. The main focus here is to teach conceptual clarity; theory can be supplemented in more advanced courses.

We will also study integrals over curves in the plane and space, and the analogous comments about choices of coordinates apply in this case. A point on a surface is described by two coordinates, a point on a curve by one coordinate.

We will always study surfaces or curves as *parametrized* in terms of one or two coordinates. Other options can be seen as special cases of this approach. For instance take a surface that is the graph of a 2-variable function f , namely the surface $z = f(x, y)$. We can describe this surface by using x, y as coordinates; or if you want to rename then into u, v for clarity, we have the parametrization $x(u, v) = u$, $y(u, v) = v$, $z(u, v) = f(u, v)$.

Similarly, if a curve in the plane is the graph of a sv function f ($y = f(x)$), this is a special case of a parametrized curve: $x(t) = t$, $y(t) = f(t)$. If a curve is the graph of a function in polar coordinates, like, e.g., the ellipse $r = 1/(1 + \varepsilon \cos \varphi)$, or the cardioid $r(\varphi) = 2(1 - \cos \varphi)$, this is also a special case of a parametrization: $r = r(\varphi)$ becomes the parametrized curve $x(\varphi) = r(\varphi) \cos \varphi$ and $y(\varphi) = r(\varphi) \sin \varphi$.

There is actually one way to describe a surface that is NOT a special case of our description in terms of coordinates: Namely, a surface could be described as the level set of a 3-variable function. In this case it's tedious to come up with a good choice of coordinates to work with. The implicit function theorem could be used to construct coordinates locally (by writing the surface as a graph locally), but these may not be the most convenient coordinates to work with. – Similar comments apply to describing a curve in the plane as level set of a 2-variable function. But this is not for you to worry now as you will not be asked to integrate over surfaces given in this implicit form.

4.2 A brief note on notation for the boundary

Let Ω be a domain in \mathbb{R}^3 . We then denote its boundary by $\partial\Omega$. This use of the curly- ∂ in the meaning of ‘boundary’ may look weird at first, but it is very common. There is no danger of misunderstanding, because we have not defined partial derivatives of *sets*, so if ∂ stands in front of a *set*, it can only mean ‘boundary’, not ‘partial’.

In a sense, the boundary of a set does represent part of a rate of change, very informally

speaking. If you have a ball and want to make its radius 1mm larger, all you need to do is to apply a 1mm thick layer of paint to its boundary. Therefore, if you take the derivative of the volume of a ball ($\frac{4}{3}\pi r^3$) with respect to the radius r , you get $4\pi r^2$, which is the surface area of the sphere, i.e., of the boundary of the ball.

Another, sneaky, reason for this notation will transpire when we study the integral theorems of vector calculus.

The same notation will be used to denote the curve $\partial\Omega$ that bounds a domain Ω in the plane, or even (if you get really advanced) on a curved surface: If Ω is the northern hemisphere, then $\partial\Omega$ would be the equator.

4.3 Definition of curve and surface integrals

If γ is a curve, say parametrized by t , we can represent a point on the curve by its position vector (vector from origin to the point) $\vec{x}(t)$, which is in \mathbb{R}^2 or \mathbb{R}^3 as the case may be. We may interpret t as a time, and γ as the path traversed by a particle over time. Then $\frac{d}{dt}\vec{x}(t)$ is the velocity of the particle. As a vector it is tangential to the curve γ , and its length $\|\vec{x}'(t)\|$ gives the speed. The integral $\int_{t_0}^{t_1} \|\vec{x}'(t)\| dt$ gives the length of the curve segment from $\vec{x}(t_0)$ to $\vec{x}(t_1)$. (The cardioid hwk problem 26 and 28 gave an example of this kind of integral.)

We can view $\|\vec{x}'(t)\| dt =: ds$ as the length element along the curve parametrized by \vec{x} . If γ is a curve representing a wire with variable ‘density’ (mass per unit length), an integral $\int_{\gamma} \rho(t) ds$ would give the mass of the wire. We never paid attention to this example because it is so natural that there is nothing new to learn.

However, we could also consider the vector length element $d\vec{s} := \vec{x}'(t)dt$. It now has a direction tangential to the curve. Now we can integrate a vector field \vec{F} along the curve (using the dot product):

$$\int_{\gamma} \vec{F} \cdot d\vec{s} := \int_{t_0}^{t_1} \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t) dt$$

This example is very important in applications: you think of \vec{F} as a force field (e.g., gravitational force depends on point in space), and we move a particle around against this force. The $\vec{F} \cdot d\vec{s}$ represents ‘a little bit of work’ done while moving the particle by the short distance $d\vec{s}$. The integral represents the total work. We will later discuss special situations in which this integral depends only on the endpoints of the curve γ , but not on γ as a whole (i.e., not on the routing between the endpoints). We can already see one thing: If the vector field \vec{F} is the gradient of a scalar function V , then

$$\int_{\gamma} \vec{\nabla}V \cdot d\vec{s} = \int_{t_0}^{t_1} \vec{\nabla}V(\vec{x}(t)) \cdot \vec{x}'(t) dt = \int_{t_0}^{t_1} \frac{d}{dt}V(\vec{x}(t)) dt = V(\vec{x}(t_1)) - V(\vec{x}(t_0))$$

So in this case the curve integral depends only on the endpoints.

As announced before, it is a straightforward application of the (single variable) chain rule to show that (in any case, gradient or not) the curve integral does not depend on the parametrization of the curve.

Next we can study integrals over surfaces. The area element of a surface in 3-space, described

by $\vec{x} = \vec{x}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}$ is given as $\|\vec{x}_u \times \vec{x}_v\| du dv$ (subscripts denoting partial derivatives). Like a fish is covered by scales, we are covering the surface with small parallelograms

whose sides are $\vec{x}_u du$ and $\vec{x}_v dv$. It is from this observation, and the knowledge about the geometric meaning of the cross product (whose norm is the area of a parallelogram) that we obtain the formula $\|\vec{x}_u \times \vec{x}_v\| du dv$ readily. So for a surface S , we define the surface integral

$$\int_S f(\vec{x}) d\text{area}(x) := \int_U f(\vec{x}(u, v)) \left\| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right\| du dv .$$

Here U is a domain in the (u, v) plane that represents the coordinates that actually occur to represent points on the surface S . For instance if S is a sphere and (u, v) are spherical coordinates with u being the angular distance from the north pole and v the geographic longitude, then U would be the rectangle $[0, \pi] \times [0, 2\pi]$.

The applications are the same as for integrals over domains: For $f \equiv 1$, we get the area of the surface. If you think of the surface representing a thin material shell, with mass density (mass per unit area) f , then $\int_S f d\text{area}$ represents the mass of the shell. With f being density times distance squared (for the distance of a point on the surface from a rotation axis), we get the moment of inertia of the surface (say of a soccer ball).

If we omit the norm around the cross product, then we get an *oriented* surface element $(\vec{x}_u \times \vec{x}_v) du dv$. Its direction is orthogonal to the surface. We have a choice of two orientations, depending on which ‘side’ of the surface the vector product is pointing. If we swap the order of the coordinates u, v , we swap the orientation, because the cross product picks up a factor -1 when we swap its factors. We will use the oriented surface element to define *flux integrals*.

Let’s say that an *oriented surface* in \mathbb{R}^3 is a surface together with a continuous choice of unit normal vector on the surface. Not every surface can be oriented. For instance the Möbius band is a surface that can be obtained if you cut out a thin long rectangle from paper, then give it a 180° twist (along the long side) and tape the short ends together. If you actually do this experiment and move along ‘one side’ of this band, you’ll see that you end up on the ‘other side’ without ever crossing the rim. So in reality, this surface does not have two sides, but only one, and it cannot be oriented.

Flux integrals make sense only over *oriented* surfaces. If the surface happens to be the boundary of a 3-dimensional domain (like the sphere as the surface of a ball), then the usual convention is to choose the orientation by letting the normal vectors point to the outside. Anybody is free to disabide by this convention, but is then under obligation to state so explicitly. If the surface does not bound a domain, there is no default choice, and the orientation needs to be stated explicitly in each case.

So, given a vector field \vec{v} in \mathbb{R}^3 and an oriented surface S , whose unit normal vector at \vec{x} is called $\vec{n}(\vec{x})$ we define (using the dot product)

$$\int_S \vec{v}(\vec{x}) \cdot \vec{n} dS := \int_S \vec{v}(\vec{x}) \cdot d\vec{S} := \int_U \vec{v}(\vec{x}(u, v)) \cdot \left(\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right) du dv .$$

The interesting part is the interpretation of this flux integral in applications. If \vec{v} is the velocity field of a fluid and $d\vec{S}$ is an oriented surface element, then $\vec{v} \cdot d\vec{S}$ describes the volume per time unit passing through this surface element. Indeed, if \vec{v} and \vec{n} are parallel (i.e., if the velocity crosses the surface orthogonally, the actual fluid passing through the surface during time dt occupies a cylinder whose basis is the surface element and whose height is $\|\vec{v}\| dt$, because all those fluid molecules that are within distance $\|\vec{v}\| dt$ from the surface will make it through the surface within time dt . So, a volume $dS \|\vec{v}\| dt$ passes through the surface element per time dt . If the angle between \vec{v} and \vec{n} is φ , then only the portion $\|\vec{v}\| \cos \varphi$ is

relevant for the fluid approaching the surface, the other component goes tangential to the surface and does therefore not contribute to crossing the surface. This cosine factor is what makes the dot product appear in this model.

Having understood this flux of volume through the surface element, it is clear that the integral represents the volume per time that flows through the entire surface.

The vector field \vec{v} might be negative the gradient of a temperature field. Heat is flowing from warm to cold, so this vector field represents the flow of heat energy. The flux integral will then represent the loss of heat energy (per time) through the surface S .

In electrostatics, a fundamental law states that the flux of the electric field through a surface $\partial\Omega$ is proportional to the total electric charge contained in the domain Ω .

We can modify this flux integral to define the flux through a curve in the plane, rather than through a surface in space. All we need is an oriented length element, but this time its orientation must not be tangential as it was in the case of the curve integral; rather it must be normal. For this reason, I will not denote it as $d\vec{s}$ to avoid confusion with the tangentially oriented length element that was so denoted, but rather write $\vec{n} ds$. But ultimately, notations vary; the notation $d\vec{s}$, which I insist on avoiding here, would be analogous to $d\vec{S}$ for the surface element, so another author may well prefer to use it. You will always need to *explain* what the symbols mean. Utter confusion will arise if you merely try to manipulate symbols. This is why in class I have often preferred word notation like *darea* over symbol notation $d(x, y)$ or dS : word notation just resists formal notation abuse more stubbornly than neat symbols.

Assume a curve γ is given in parametrized form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Now since the tangential $d\vec{s}$ is given by $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} dt$, the normal length element $\vec{n}ds$ can only be one of the following two: $\pm \begin{bmatrix} -y'(t) \\ x'(t) \end{bmatrix} dt$. Which of the two it is depends on the orientation you choose. These two are determined by having the same length as $d\vec{s}$ and being orthogonal to it.

If you resent the approach of just exhibiting the result out of the blue and saying “see that’s it because it fulfills all requirements” (as we have just done here), you could think of the curve in the plane as the base of a cylindrical surface in space: $x(t, u) := x(t)$, $y(t, u) := y(t)$, $z(t, u) := u$. Then use the oriented surface element (calculated using a cross product). It has a horizontal direction. Drop the z component 0, divide the area element by the height du and get the length element.

5 Vectorial derivative operations

5.1 Nabla and Gradient

The symbol ∇ denotes a ‘vector valued differential operator’, called ‘nabla’. For more emphasis of its vector nature it can also be written as $\vec{\nabla}$. ($\vec{\nabla}$ and ∇ mean the very same thing.) In components (with respect to a cartesian coordinate system), it is written as

$$\vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

(or the two-component analog, if we are working in 2 dimensions).

If you ‘multiply’ $\vec{\nabla}$ with a scalar valued function f the way how you multiply a vector with a scalar (namely each component of the vector gets multiplied by that scalar), you get the gradient of f :

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} f(x, y, z) \\ \frac{\partial}{\partial y} f(x, y, z) \\ \frac{\partial}{\partial z} f(x, y, z) \end{bmatrix}$$

The ‘multiplication’ is to be understood symbolically, because the partial derivative of f is not actually a product of $\frac{\partial}{\partial x}$ and f , it is only written like a product.

We had written ∇f (or $\vec{\nabla} f$) for the gradient early in the semester, but now we are looking at the object $\vec{\nabla}$ in its own right and want to see what other mischief we can do with it.

5.2 Divergence of a vector field

We can take a (symbolic) dot product of $\vec{\nabla}$ with a vector field $\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$. Then we get

$$\vec{\nabla} \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} =: \operatorname{div} f.$$

The resulting *scalar* field is called the divergence of \vec{f} . The word ‘divergence’ has nothing to do with convergent or divergent series. The geometric meaning of the divergence of a vector field can only be understood together with Gauss’ integral theorem, also known as divergence theorem, which will be explained below. It will turn out, as a consequence of this theorem, that

$$\operatorname{div} \vec{f}(\vec{x}) = \lim \frac{\int_{\partial\Omega} \vec{f}(\vec{x}) \cdot d\vec{S}}{\operatorname{vol}(\Omega)},$$

where the limit refers to a sequence of domains Ω shrinking to the point \vec{x} . For instance one could take $\Omega := B_r(\vec{x})$, a ball of radius r about \vec{x} , and then the limit would be taken as $r \rightarrow 0$.

If \vec{v} is a velocity field of some fluid, then $\operatorname{div} \vec{v}$ denotes a ‘volume production rate density’. For an incompressible fluid (the volume cannot change), the flux integral over a surface $\partial\Omega$ must be 0: as much as flows in, must flow out again. In this case $\operatorname{div} \vec{v} = 0$. On the other hand, if \vec{v} denotes the velocity field of a gas (which can expand or be compressed, due to changes in pressure and or temperature), then $\operatorname{div} \vec{v}$ may be nonzero. The gas expands, where the divergence is positive, but gets compressed where the divergence is negative. This explains the choice of the word ‘divergence’ because the Latin root in ‘diverge’ means to ‘go apart’.

5.3 Curl of a vector field

We also can take a symbolic cross product with $\vec{\nabla}$. This of course works only in 3 dimensions, unlike the dot product, which is defined in any dimension. We get

$$\vec{\nabla} \times \vec{f} = \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix} =: \operatorname{curl} \vec{f}$$

Again it takes an integral theorem of vector calculus outlined below (Stokes' theorem) to understand the geometric meaning of the curl of a vector field. The upshot is, for a velocity field \vec{v} , that its curl describes whether the fluid rotates *locally*. But beware! It may sometimes look, when you glance at the drawing of a vector field, like fluid was rotating (because indeed it goes around in circles on a *global* scale), but that doesn't mean it rotates *locally*. (Marsden-Tromba give an illustrative picture of this phenomenon, on which I don't want to dwell at the moment).

While we cannot, strictly speaking, take the curl of a vector field in \mathbb{R}^2 , we can treat such a vector field as a vector field in \mathbb{R}^3 : We just convert $\begin{bmatrix} v_1(x, y) \\ v_2(x, y) \end{bmatrix}$ into $\begin{bmatrix} v_1(x, y) \\ v_2(x, y) \\ 0 \end{bmatrix}$, a vector field whose components 'happen to be' independent of the 3rd coordinate z . The curl of this vector field has only a z component, and it is this (scalar) quantity that gives the local rotation information of the curl. This observation allows us to use 2-dimensional examples as an illustration of curl if we so choose.

6 Integral theorems of vector calculus

We have basically two such theorems: Gauss' theorem (which is related to the divergence) and works in any dimension (specifically 2 and 3, which interest us most), and Stokes' theorem (which is related to the curl) and works only in 3 dimensions (but could be patched into 2 dimensions by the embedding trick we have just mentioned).

Both of these integral theorems generalize the fundamental theorem of calculus: They reduce an integral of a derivative of a function over some domain into an integral of the function itself over the (oriented) boundary of that domain.

In sv calculus, we have

$$\int_{[a,b]} f'(x) dx = f(b) - f(a)$$

The domain is the interval $[a, b]$, its boundary consists of only the two points a and b , and the 'integral' over this boundary is simply a sum of two terms. But wait a minute, we should be talking about an *oriented* boundary. So if you want to look to the outside of $[a, b]$ at a , you look in the negative direction, whereas at b , you'd look in the positive direction. This is represented by the fact that we have $f(b) - f(a)$ and not $f(b) + f(a)$ on the right side.

The more advanced formalism of differential forms would re-unite Gauss' and Stokes' theorem under one heading, and it would also remove the dimension restriction for the curl, and Stokes' theorem. But we're not going there in this class.

6.1 Gauss' theorem

Gauss' theorem says:

$$\int_{\Omega} \operatorname{div} \vec{f}(\vec{x}) d\operatorname{vol} = \int_{\partial\Omega} \vec{f} \cdot \vec{n} dS$$

Here the volume integral of a divergence equals a flux integral of the vector field itself over the boundary. \vec{n} is the *exterior* unit normal, in agreement with the convention we made concerning the orientation of a boundary. The analogous integral theorem holds in 2 dimensions.

The limit property of the divergence mentioned above is an immediate consequence of Gauss' theorem.

The proof of Gauss' theorem is interesting: It first suffices to prove the theorem for vector fields of the forms $\begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}$. The general case can be obtained by adding the results for the special cases. Next it suffices to prove the theorem for special domains Ω , say of the form $(x, y) \in U$, $\psi_1(x, y) \leq z \leq \psi_2(x, y)$. Reasonably well-behaved general domains can be pieced together from these special domains. The 'inner boundaries' (boundaries of the pieces that are inside Ω) give rise to surface flux integrals that cancel each other out when we add the pieces together. If we prove the theorem for these simple pieces, the theorem reduces to the fundamental theorem of sv calculus, by explicit calculation.

There are a few awkward questions hidden here, like: which domains can be pieced together in the required way? A more advanced approach would bypass this issue by using a different way of cutting the domain apart, an approach that is much more versatile for vastly arbitrary domains. Again this is better left to more advanced courses.

6.2 Stokes' theorem

Stokes' theorem relates the flux integral of a curl with a curve integral over the boundary of the surface.

Suppose S is a surface in \mathbb{R}^3 with boundary ∂S . This boundary is then a closed curve (or it may consist of several curves, if the surface has 'holes'), and the orientation of this (these) curve(s) is related to the orientation of the boundary via the right hand rule as follows: If the four fingers of the right hand point along the curve in the direction of increasing parameter, with the palm facing the surface, then the thumb determines the normals on the surface. In other words, if you walk along the boundary of the surface, with the surface on your left, then the normal vectors on the surface point 'upwards'.

With these conventions, Stokes' theorem states:

$$\int_S \text{curl} \vec{v}(\vec{x}) \cdot d\vec{S} = \int_{\partial S} \vec{v} \cdot d\vec{s}$$

Stokes' theorem has some interesting consequences:

Note that in \mathbb{R}^3 , for any given closed curve γ (e.g., a circle), there are many surfaces S whose boundary is γ . For instance, with γ a circle, you can take for S either a disk, or a hemisphere that makes the circle into an equator, or you have many other options. The flux integral on the left must be the same, regardless which surface you choose, because the right hand side depends only on the boundary of that surface.

Let's look at an example in this style: We consider the unit ball B in \mathbb{R}^3 and assume the north pole to be at $(0, 0, 1)$. Take S_+ to be the northern hemisphere, S_0 a disk in the equator plane, and S_- the southern hemisphere. Each of these has the equator as boundary. The orientation of the equator is chosen to be in eastward direction. This means that the normal vector field on each of S_+ , S_- , S_0 points somewhat 'upwards' (with positive z component). Now S_+ together with either S_0 or S_- makes up the boundary of either a half-ball or a full ball. However, in order to have the 'correct' orientation of the normal to point outward, we need to change the orientation we had on S_- or S_0 . This means their boundary has to swap orientation, too: from their point of view, the equator has to be traversed westward. Now

if we add the two versions of Stokes' theorem:

$$\begin{aligned} \int_{S_+} \text{curl} \vec{v}(\vec{x}) \cdot d\vec{S} &= \int_{\partial S_+} \vec{v} \cdot d\vec{s} \\ \int_{S_-} \text{curl} \vec{v}(\vec{x}) \cdot d\vec{S} &= \int_{\partial S_-} \vec{v} \cdot d\vec{s} \end{aligned}$$

the right hand sides cancel out because of the opposite orientation, and the left hand sides add up to an integral over ∂B . We can now use Gauss' theorem to conclude that

$$0 = \int_{\partial B} \text{curl} \vec{v} \cdot d\vec{S} = \int_B \text{div} \text{curl} \vec{v} \, d\text{vol} .$$

How can this be true for an arbitrary vector field \vec{v} ? Easy: You can calculate explicitly that

Theorem: $\text{div} \text{curl} \vec{v} = 0$

for every C^2 vector field \vec{v} . Write it out in components and see that all second partial derivatives cancel each other! (The theorem that partial derivatives may be calculated in any order enters here.) — There is also a suggestive ‘proof’ of this result using the symbolic $\vec{\nabla}$ vector: $\text{div} \text{curl} \vec{v} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v})$. If two vectors in a scalar triple product are the same, then the scalar triple product is 0. However, I would not call this a genuine proof, because this is not a true scalar triple product, but only a symbolic one. It is only a suggestion for a proof, and needs independent confirmation that the algebra carries over.

If you ever try to grab a formula for $\vec{a} \times (\vec{b} \times \vec{c})$ from a book on vector algebra and then use it ruthlessly to get a formula for the symbolic cross product $\vec{\nabla} \times (\vec{u} \times \vec{v})$ you'll likely get a wrong result, and thus see that my warning about confirming the suggested proof was well placed.

There is another formula that is easy to check quickly by writing explicit components:

Theorem: $\text{curl} \vec{\nabla} V = 0$

for every C^2 scalar field V ; in other words, the curl of a gradient vanishes.

We had seen, when we studied curve integrals, that a curve integral $\int_{\gamma} \vec{F} \cdot d\vec{s}$ depends only on the endpoints of γ , provided $\vec{F} = \vec{\nabla} V$. Stokes' theorem confirms this: The difference between two curve integrals $\int_{\gamma_1} \vec{F} \cdot d\vec{s}$ and $\int_{\gamma_2} \vec{F} \cdot d\vec{s}$ with the same endpoints is the curve integral over a closed curve γ_0 . If we take any surface S whose boundary is this closed curve γ_0 , then the integral $\int_{\gamma_0} \vec{F} \cdot d\vec{s}$ is by Stokes the flux integral of a curl over that surface; and it is zero if the curl of the vector field \vec{F} vanishes.

But now we can turn the argument around and say: If the curl of a vector field vanishes, then the curve integral of this vector field depends only on the endpoints of the curve, *provided* we can span some surface between any two curves with the same endpoints, such that we can use this surface for Stokes' theorem. Can we span such a surface? The answer is (intuitively) ‘yes’ if the vector field \vec{F} is defined in all of \mathbb{R}^3 . However, if the vector field is defined on a domain with ‘holes’ as for instance the vector field $\vec{F}(x, y, z) = [\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 0]^T$ (which is not defined for $x = y = 0$, i.e., on the z axis), then we cannot span a surface that bounds the circle $x^2 + y^2 = 1$, except we let this surface cross the z axis. But with \vec{F} undefined on the z -axis, we cannot use Stokes' theorem and can therefore not conclude that $\int \vec{F} \cdot d\vec{s} = 0$ over the unit circle. It is not, even though $\text{curl} \vec{F} = \vec{0}$.

(BTW, This example vector field I have given here represents the magnetic field around a wire along the z axis, through which an electric current flows.)

THE END