

Hwk #52:

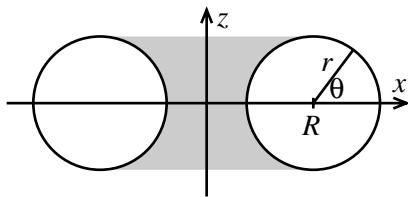
The torus T arises by rotating the disc with radius r centered at $(x, z) = (R, 0)$ about the z axis. Here we assume that $R > r$. Draw a figure. Choose coordinates on the surface of the torus. (Hint: take two angles, one being the rotation angle, the other describing the location of a point on the circle that is to be rotated.)

Calculate the flux of the vector field $[xy^2, yx^2, 1]^T$ through the surface ∂T of the torus T directly by the definition.

Calculate the same quantity as a volume integral by using the divergence theorem. (Hint: For the volume integral, you may consider using cylinder coordinates, or else coordinates that are specially adjusted to the solid torus, in analogy to the angle coordinates you should have used on its surface.)

Solution: The figure shows a cross section of the torus.

6 pts



The circle on the right can be parametrized as

$$\begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} = \begin{bmatrix} R + r \cos \theta \\ r \sin \theta \end{bmatrix}$$

As we rotate this circle around the z axis, the x coordinate of the cross section becomes a radius in the xy plane (call it ρ , since we have hogged up r and R already), and the x, y of the rotated body are now $\rho \cos \phi$ and $\rho \sin \phi$ respectively. For r (and R) fixed, we can parametrize the surface of the torus by

$$\begin{bmatrix} x(\theta, \phi) \\ y(\theta, \phi) \\ z(\theta, \phi) \end{bmatrix} = \begin{bmatrix} (R + r \cos \theta) \cos \phi \\ (R + r \cos \theta) \sin \phi \\ r \sin \theta \end{bmatrix}, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq 2\pi$$

The oriented surface element is $d\vec{S} = \frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial \theta} d\phi d\theta$, i.e.,

$$d\vec{S} = \left(\begin{bmatrix} -(R + r \cos \theta) \sin \phi \\ (R + r \cos \theta) \cos \phi \\ 0 \end{bmatrix} \times \begin{bmatrix} -r \sin \theta \cos \phi \\ -r \sin \theta \sin \phi \\ r \cos \theta \end{bmatrix} \right) d\phi d\theta = \begin{bmatrix} (R + r \cos \theta)r \cos \theta \cos \phi \\ (R + r \cos \theta)r \cos \theta \sin \phi \\ (R + r \cos \theta)r \sin \theta \end{bmatrix} d\phi d\theta$$

(This order of factors in the cross product gives the desired orientation outward.) The flux integral can now be evaluated:

$$\begin{aligned} \Phi &:= \int_{\partial T} \begin{bmatrix} xy^2 \\ yx^2 \\ 1 \end{bmatrix} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{2\pi} \begin{bmatrix} (R + r \cos \theta)^3 \cos \phi \sin^2 \phi \\ (R + r \cos \theta)^3 \sin \phi \cos^2 \phi \\ 1 \end{bmatrix} \cdot \begin{bmatrix} (R + r \cos \theta)r \cos \theta \cos \phi \\ (R + r \cos \theta)r \cos \theta \sin \phi \\ (R + r \cos \theta)r \sin \theta \end{bmatrix} d\theta d\phi \\ &= r \int_0^{2\pi} \int_0^{2\pi} \left(2(R + r \cos \theta)^4 \cos^2 \phi \sin^2 \phi \cos \theta + (R + r \cos \theta) \sin \theta \right) d\phi d\theta \end{aligned}$$

The ϕ integral is easiest to evaluate. Note that $\sin^2 \phi \cos^2 \phi = \frac{1}{4} \sin^2(2\phi)$, which integrates to $\frac{1}{8} 2\pi$ over $[0, 2\pi]$. So

$$\Phi = 2\pi r \int_0^{2\pi} \frac{2}{8} (R + r \cos \theta)^4 \cos \theta d\theta + 2\pi r \int_0^{2\pi} (R + r \cos \theta) \sin \theta d\theta$$

The second integral averages out to 0. In the first integral, it is probably best to expand the 4th power and note that odd powers of $\cos \theta$ will also average out to 0 when integrated. So we are left with

$$\Phi = \frac{\pi r}{2} \int_0^{2\pi} (4R^3 r \cos^2 \theta + 4Rr^3 \cos^4 \theta) d\theta = 2\pi^2 R^3 r^2 + \frac{3}{2}\pi^2 Rr^4$$

Now the divergence of our vector field is $\partial_x(xy^2) + \partial_y(x^2y) + \partial_z(1) = x^2 + y^2$. By the divergence theorem, $\Phi = \int_T (x^2 + y^2) d\text{vol}$. (Incidentally, this quantity is a moment of inertia. I thought when I chase you through a calculation, I might just as well choose one whose result is meaningful. If you know the formula for the volume of the torus, or calculate it similarly as we are going to calculate the volume integral, you see that $\Phi = m(R^2 + \frac{3}{4}r^2)$.)

Now let's tackle the volume integral directly and check that it evaluates to the same quantity Φ , as it should. We can use cylinder coordinates. Remember that we call the radius ρ this time, because we have hogged up R and r already otherwise. Now $d\text{vol} = \rho d\rho d\phi dz$. The limits of integration in z direction need some attention. ρ can only go from $R - r$ to $R + r$, and for each ρ in this interval, z is to be integrated over such an interval that $z^2 + (R - \rho)^2 \leq r^2$. So we conclude

$$I := \int_T (x^2 + y^2) d\text{vol} = \int_{R-r}^{R+r} \int_{-\sqrt{r^2 - (R-\rho)^2}}^{\sqrt{r^2 - (R-\rho)^2}} \int_0^{2\pi} \rho^2 \rho d\phi dz d\rho = 4\pi \int_{R-r}^{R+r} \rho^3 \sqrt{r^2 - (R-\rho)^2} d\rho$$

Substituting $\rho = R + rt$, $d\rho = r dt$, simplifies the root some:

$$I = 4\pi r^2 \int_{-1}^1 (R + rt)^3 \sqrt{1 - t^2} dt$$

The trig substitution $t = \cos \theta$ (with the θ indeed denoting the same quantity as before), and expansion of the cube (with odd powers of t leading to 0 integrals) gives:

$$I = 4\pi r^2 \int_0^\pi (R^3 + 3Rr^2 \cos^2 \theta) \sin^2 \theta d\theta = 2\pi^2 r^2 \left(R^3 + \frac{3}{4}Rr^2 \right)$$

It may also be interesting to use the same 'taylor-made' torus coordinates for the evaluation of the volume integral. Only this time, the radial coordinate in the circle is not the fixed r , but is a variable that runs from 0 to r . So we now have

$$\begin{bmatrix} x(l, \theta, \phi) \\ y(l, \theta, \phi) \\ z(l, \theta, \phi) \end{bmatrix} = \begin{bmatrix} (R + l \cos \theta) \cos \phi \\ (R + l \cos \theta) \sin \phi \\ l \sin \theta \end{bmatrix}, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq l \leq r$$

The volume element requires the evaluation of a 3×3 determinant:

$$\det \begin{bmatrix} x_l & x_\theta & x_\phi \\ y_l & y_\theta & y_\phi \\ z_l & z_\theta & z_\phi \end{bmatrix} = \dots = -l(R + l \cos \theta)$$

So $d\text{vol} = l(R + l \cos \theta) dl d\theta d\phi$. This is also to be expected geometrically: The length element along a 'circle of latitude' is $\rho d\phi = (R + l \cos \theta) d\phi$. The length element along a 'meridian' is $l d\theta$. They form an approximate rectangle whose area is the product. On these we erect a box of height dl .

Now we calculate in these coordinates

$$I = \int_T (x^2 + y^2) d\text{vol} = \int_0^r \int_0^{2\pi} \int_0^{2\pi} (R + l \cos \theta)^2 l (R + l \cos \theta) d\phi d\theta dl$$

The evaluation is similar as before, and it confirms the result.

Hwk #53:

Suppose you have a simple closed parametrized curve $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $0 \leq t \leq T$, in the plane. Here ‘simple’ means that the curve does not intersect itself; closed means that the point at $t = 0$ is the same as the one at $t = T$. Suppose that this curve bounds a domain, which we call Ω . (BTW: A simple closed curve in the plane always bounds some domain; you may find that intuitively obvious, even though it’s surprisingly difficult to prove.) We will also assume that if we move through the curve with increasing t , the domain lies to the left. \vec{n} is meant to be exterior unit normal vector. The components of a vector field \vec{f} will be called f_1, f_2 , i.e., $\vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Write the 2-dimensional flux integral $\int_{\partial\Omega} \vec{f}(x, y) \cdot \vec{n} ds$ explicitly as \int_0^T (find out what) dt .

Now use in particular the vector field $\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}$ in combination with the divergence theorem to obtain a formula for the area of Ω in terms of the curve $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ that bounds it.

Solution: The tangential length element is $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} dt$. So turning this vector by 90 degrees, we see 4 pts that $\vec{n} ds = \pm \begin{bmatrix} y'(t) \\ -x'(t) \end{bmatrix} dt$. If we mean to turn it counterclockwise, we have to choose the + sign. This is the appropriate choice if the domain lies to the left of the curve and we want an outwards pointing normal vector. So the flux integral becomes:

$$\int_{\gamma} \vec{f} \cdot \vec{n} ds = \int_{t_0}^{t_1} \{ f_1(x(t), y(t))y'(t) - f_2(x(t), y(t))x'(t) \} dt$$

For the specific vector field given, its divergence is 1, and the divergence theorem turns into the following neat formula:

$$\text{area}(\Omega) = \frac{1}{2} \int_{\partial\Omega} (x(t)y'(t) - y(t)x'(t)) dt$$

Hwk #54:

Let’s modify the cardioid from Hwk #26: The point P is no longer on the boundary of the circle C_2 , but in the interior, a distance $a < 1$ away from the center. We again consider the curve traced out by P . This time we only have the parametric representation of the boundary, but not a readily available formula $r = r(\varphi)$ in polar coordinates. You may assume that the curve has no self-intersections (it can be proved, but we won’t bother doing it).

Now calculate the area enclosed by this modified cardioid, using the formula from the previous problem. (You may then check that the case $a = 0$ gives the result that it obviously ought to give, namely what? Such a check is a small insurance against calculational errors.)

Solution: We need a parametrization of the (generalized) cardioid: Comparing with problem number 26, we find (by means of the same calculation, with just trivial modifications):

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 + 2 \cos t - a \cos 2t \\ 2 \sin t - a \sin 2t \end{bmatrix}$$

Then $\frac{1}{2}(xy' - yx') = 2 + a^2 - (1 + 3a) \cos t + a \cos 2t$. Integrating this over $t \in [0, 2\pi]$ gives the area $2\pi(2 + a^2)$. We can confirm this result for $a = 1$ from the old cardioid problem, and for $a = 0$, because in this case we merely get a circle of radius 2. It is also appropriate that this result remains unchanged if we replace a with $-a$: this change amounts to putting the tracing point on the other side of the center of the moving circle, which just produces a rotated version of the same cardioid, so it should have the same area.

Hwk #55:

Consider the vector field $\vec{F}(x, y, z) = \begin{bmatrix} x + y \\ y + z \\ z + x \end{bmatrix}$ and the following curves:

γ_1 is the straight line segment connecting $(0, 0, 0)$ to $(1, 1, 1)$. γ_2 is the path consisting of three straight line segments, going from $(0, 0, 0)$ to $(0, 0, 1)$, then to $(0, 1, 1)$ and finally to $(1, 1, 1)$.

Now calculate the curve integrals $\int_{\gamma_1} \vec{F} \cdot d\vec{s}$ and $\int_{\gamma_2} \vec{F} \cdot d\vec{s}$. Here $d\vec{s}$ is the *tangentially oriented* length element. Are they the same?

Solution: It's easy to parametrize γ_1 :

5 pts

$$\vec{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \quad 0 \leq t \leq 1$$

So

$$\int_{\gamma_1} \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t) dt = \int_0^1 (2t + 2t + 2t) dt = [3t^2]_0^1 = 3$$

Curve γ_2 is best decomposed in three pieces, each of which is parametrized separately:

$$\gamma_{2a} : \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}, \quad 0 \leq t \leq 1 \quad \gamma_{2b} : \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}, \quad 0 \leq t \leq 1 \quad \gamma_{2c} : \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix}, \quad 0 \leq t \leq 1$$

So

$$\int_{\gamma_2} \vec{F} \cdot d\vec{s} = \int_0^1 (t + 0) 1 dt + \int_0^1 (t + 1) 1 dt + \int_0^1 (t + 1) 1 dt = \frac{1}{2} + \frac{3}{2} + \frac{3}{2} = \frac{7}{2}$$

So the two paths give different integrals, as they well may, since the curl of the vector field is not 0.