Hwk #52:

The torus T arises by rotating the disc with radius r centered at $(x, z) = (R, 0)$ about the z axis. Here we assume that $R > r$. Draw a figure. Choose coordinates on the surface of the torus. (Hint: take two angles, one being the rotation angle, the other describing the location of a point on the circle that is to be rotated.)

Calculate the flux of the vector field $[xy^2,\, yx^2,\, 1]^T$ through the surface ∂T of the torus T directly by the definition.

Calculate the same quantity as a volume integral by using the divergence theorem. (Hint: For the volume integral, you may consider using cylinder coordinates, or else coordinates that are specially adjusted to the solid torus, in analogy to the angle coordinates you should have used on its surface.)

Hwk #53:

Suppose you have a simple closed parametrized curve $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $0 \le t \le T$, in the plane. Here 'simple' means that the curve does not intersect itself; closed means that the point at $t = 0$ is the same as the one at $t = T$. Suppose that this curve bounds a domain, which we call Ω. (BTW: A simple closed curve in the plane always bounds some domain; you may find that intuitively obvious, even though it's surprisingly difficult to prove.) We will also assume that if we move through the curve with increasing t, the domain lies to the left. $\vec{n}$ is meant to be exterior unit normal vector. The components of a vector field $\vec{f}$ will be called f_1, f_2, i.e., $\vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Write the 2-dimensional flux integral $\int_{\partial\Omega} \vec{f}(x, y) \cdot \vec{n} ds$ explicitly as $\int_0^T (\text{find out what})\, dt$.

Now use in particular the vector field $\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}$ in combination with the divergence theorem to obtain a formula for the area of Ω in terms of the curve $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ that bounds it.

Hwk #54:

Let's modify the cardioid from Hwk #26: The point P is no longer on the boundary of the circle C_2, but in the interior, a distance $a < 1$ away from the center. We again consider the curve tranced out by P. This time we only have the parametric representation of the boundary, but not a readily available formula $r = r(\varphi)$ in polar coordinates. You may assume that the curve has no self-intersections (it can be proved, but we won't bother doing it).

Now calculate the area enclosed by this modified cardioid, using the formula from the previous problem. (You may then check that the case $a = 0$ gives the result that it obviously ought to give, namely what? Such a check is a small insurance against calculational errors.)

Hwk #55:

Consider the vector field $\vec{F}(x, y, z) = \begin{bmatrix} x+y \\ y+z \\ z+x \end{bmatrix}$ and the following curves:

γ_1 is the straight line segment connecting $(0, 0, 0)$ to $(1, 1, 1)$. γ_2 is the path consisting of three straight line segments, going from $(0, 0, 0)$ to $(0, 0, 1)$, then to $(0, 1, 1)$ and finally to $(1, 1, 1)$.

Now calculate the curve integrals $\int_{\gamma_1} \vec{F} \cdot d\vec{s}$ and $\int_{\gamma_2} \vec{F} \cdot d\vec{s}$. Here $d\vec{s}$ is the *tangentially oriented* length element. Are they the same?