Hwk #47:

Calculate the Jacobi matrix (total derivative) of the coordinate transformation

$$\begin{aligned} x &= x(r, \vartheta, \varphi) = r \sin\vartheta \, \cos\varphi \\ y &= y(r, \vartheta, \varphi) = r \sin\vartheta \, \sin\varphi \\ z &= z(r, \vartheta, \varphi) = r \cos\vartheta \end{aligned}$$

and obtain the volume element in these (spherical) coordinates. Also for fixed r obtain the surface area element.

Use these to calculate the volume of a ball of radius R, and its surface area, directly from the multivariable integral formalism. *Of course you know these formulas already and can also obtain them from single variable integral methods applied to rotation surfaces and bodies. But here the punchline is that you use the newly acquired methods on this simple test case.*

Solution: 5 pts

$$\begin{bmatrix} \partial x/\partial r & \partial x/\partial \vartheta & \partial x/\partial \varphi \\ \partial y/\partial r & \partial y/\partial \vartheta & \partial y/\partial \varphi \\ \partial z/\partial r & \partial z/\partial \vartheta & \partial z/\partial \varphi \end{bmatrix} = \begin{bmatrix} \sin\vartheta\,\cos\varphi & r\cos\vartheta\,\cos\varphi & -r\sin\vartheta\,\sin\varphi \\ \sin\vartheta\,\sin\varphi & r\cos\vartheta\,\sin\varphi & r\sin\vartheta\,\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{bmatrix}$$

The determinant of this matrix is

$$r^2\Big(\sin\vartheta\,\cos^2\vartheta\,(\cos^2\varphi + \sin^2\varphi) + \sin^3\vartheta\,(\sin^2\varphi + \cos^2\varphi)\Big) = r^2 \sin\vartheta$$

So the volume element is $r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi$.

For the surface area element we need to calculate

$$\begin{bmatrix} r\cos\vartheta\,\cos\varphi \\ r\cos\vartheta\,\sin\varphi \\ -r\sin\vartheta \end{bmatrix} \times \begin{bmatrix} -r\sin\vartheta\,\sin\varphi \\ r\sin\vartheta\,\cos\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} r^2\sin^2\vartheta\,\cos\varphi \\ r^2\sin^2\vartheta\,\sin\varphi \\ r^2\sin\vartheta\,\cos\vartheta \end{bmatrix}$$

The norm of this latter vector is $r^2 \sin\vartheta$ (using $\theta \in [0, \pi]$, because otherwise we would get $|\sin\vartheta|$ instead). So the surface area element is $r^2 \sin\vartheta \, d\vartheta \, d\varphi$.

Note as an aside: This is of course to be expected: since the radial direction is orthogonal to the surface area (the sphere), the volume element is dr times the surface area element.

Now the volume of the ball of radius R is

$$V = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\vartheta \, d\varphi \, d\vartheta \, dr = \int_0^R \int_0^\pi 2\pi r^2 \sin\vartheta \, d\vartheta \, dr = 2\pi \int_0^R 2r^2 \, dr = \frac{4\pi}{3} R^3$$

The same calculation, omitting the outermost $\int_0^R \ldots dr$ and the last step, gives the surface area $4\pi r^2$ for a sphere of radius r.

Hwk #48:

The curve $y = f(x)$ $(x_0 \le x \le x_1)$ rotates about the x axis. Obtain the formula from single variable calculus as a special case from the area formula for general surfaces.

Solution: We can use x and the rotation angle ϕ as coordinates. To avoid confusion, we rename x into t when it is one of two coordinates, but leave it as x when it is one of three cartesian coordinates. (This renaming is not necessary, and you may disagree about its desirability.) 5 pts

So we have $x = t$, $y = f(t)\cos\phi$, $z = f(t)\sin\phi$. *(Note that these (x, y, z) represent a point on the rotation surface. In particular y is not equal to $f(t)$ everywhere on the surface (even though these two are equal on the curve whose rotation sweeps out the surface).*

Writing $\vec{x}$ for the vector $[x,\, y,\, z]^T$, we note that $d\text{area} = \|\vec{x}_t \times \vec{x}_\phi\|\, dt\, d\phi$. Calculate

$$\vec{x}_t \times \vec{x}_\phi = \begin{bmatrix} 1 \\ f'(t)\cos\phi \\ f'(t)\sin\phi \end{bmatrix} \times \begin{bmatrix} 0 \\ -f(t)\sin\phi \\ f(t)\cos\phi \end{bmatrix} = \begin{bmatrix} f(t)f'(t) \\ -f(t)\cos\phi \\ -f(t)\sin\phi \end{bmatrix}$$

$$\|\vec{x}_t \times \vec{x}_\phi\| = \sqrt{f(t)^2 f'(t)^2 + f(t)^2} = f(t)\sqrt{1 + f'(t)^2}$$

Integrating this over the surface gives $\int_{x_0}^{x_1} \int_0^{2\pi} f(t)\sqrt{1 + f'(t)^2}\, d\phi\, dt = 2\pi \int_{x_0}^{x_1} f(t)\sqrt{1 + f'(t)^2}\, dt$, as expected from Calculus 2.

Hwk #49:

Physicists are familiar with the following phenomenon: If you let a massive ball and a massive cylinder roll down an incline, then the ball rolls more rapidly than the cylinder. The reason is that part of the potential energy gained when losing height is converted into kinetic energy for the forward movment, whereas another part is converted into 'internal' (rotational) kinetic energy, because the object is rolling rather than just sliding. This rotational energy is lost to the forward motion.

You may know the formula $\frac{1}{2}mv^2$ (half mass times velocity^2) for the translation energy. There is a similar formula $\frac{1}{2}I\omega^2$ for the rotation energy, where ω measures how many radians per time unit an object rotates. The quantity I is called 'moment of inertia' and it depends on the mass distribution in the body. Mass that is closer to the rotation axis counts less because it does not move as fast as mass that is farther away from the rotation axis.

The formula for I is: $I = \int_{\text{body}} s^2 \rho d\text{vol}(x, y, z)$. Here ρ is the density (which may depend on (x, y, z), but in this problem we assume it is constant). s denotes the distance from the rotation axis, which you have to express in terms of x, y, z or whatever coordinates you use.

Given this wisdom, I ask you to find I for a cylinder of radius R and height h, and also for a ball of radius R. In either case, these objects rotate about a symmetry axis. You are to express the result in the form: number times (total mass) times R^2. Remember that the total mass is volume times density ρ.

The larger the number in front of 'mass times R^2' is, the higher the proportion of energy that is used for the rotation.

Solution: If we let the ball rotate about the z axis (or rather we may say: if we choose the z axis to be the axis of rotation of the ball) and use the sherical coordiantes from Hwk #47, we note that $s = r\sin\vartheta$. So we get (with the substitution $t = -\cos\vartheta$ near the end) 5 pts

$$I = \rho\int_{\text{ball}} r^2\sin^2\vartheta\,d\text{vol} = \rho\int_0^R\int_0^\pi\int_0^{2\pi}(r^2\sin^2\vartheta)\,(r^2\sin\vartheta)\,d\varphi\,d\vartheta\,dr = 2\pi\rho\int_0^R\int_0^\pi r^4\sin^3\vartheta\,d\vartheta\,dr$$
$$= 2\pi\rho\frac{R^5}{5}\int_0^\pi\sin^3\vartheta\,d\vartheta = \frac{2\pi\rho\,R^5}{5}\int_{-1}^1(1-t^2)\,dt = \frac{2\pi\rho\,R^5}{5}\,\frac{4}{3} = \frac{2}{5}(\text{mass})R^2$$

Now for the analogous calculation for a cylinder of radius R and height h, we put the z axis along the axis of the cylinder, and we let, for instance, run z from 0 to h. (Other choices like $z \in [-\frac{h}{2}, \frac{h}{2}]$ would be justas good and lead to the same result.) We use polar coordinates in the plane, and z as a third coordinate (remember that we called these cylindrical coordinates). So we have $x = r\cos\phi$, $y = r\sin\phi$, $z = z$, and $d\text{vol} = r\,dr\,d\phi\,dz$. The distance s from the axis of rotation is r. So we calculate

$$I = \rho\int_0^R\int_0^{2\pi}\int_0^h r^2\cdot r\,dz\,d\phi\,dr = \rho\frac{R^4}{4}\cdot 2\pi\cdot h = \frac{1}{2}(\text{mass})R^2$$

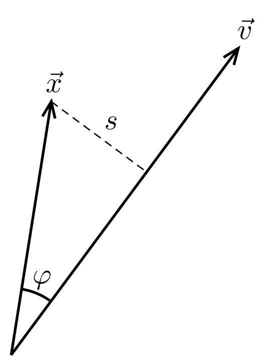
Hwk #50:

Now we rotate a cube $-a \le x, y, z \le a$ about an axis through the origin. The axis goes in the direction of a vector $\vec{v}$.

First draw a generic picture of a vector $\vec{x} = [x,\, y,\, z]^T$ and a vector $\vec{v} = [v_1,\, v_2,\, v_3]^T$ (both starting at the origin) and find a formula for the distance s of the tip of $\vec{x}$ (i.e., of the point (x, y, z)) from the axis that goes along the vector $\vec{v}$.

Then calculate the moment of inertia for this rotation (expressed as number times mass times a^2). Surprise: The final result will not depend on $\vec{v}$ — (To those who know about the tensor of inertia and the role eigenvalues play there, this surprise will be expected; but these wise folks, that's not us, for the time of Calc 3.)

Solution: Consider the following picture, in which the vectors begin in the origin: 5 pts



$\vec{v}$ points along the axis of rotation, and we want the distance s of a point represented by its position vector $\vec{x}$ from this axis.

$$s^2 = \|\vec{x}\|^2\sin^2\varphi = \|\vec{x}\|^2 - \|\vec{x}\|^2\cos^2\varphi = \|\vec{x}\|^2 - \Big((\vec{x}\cdot\vec{v})/\|\vec{v}\|\Big)^2$$

If we write $\vec{x} =: [x,\, y,\, z]^T$ and $\vec{v} =: [u,\, v,\, w]^T$ in components, we get

$$s^2 = (x^2 + y^2 + z^2) - \frac{(xu + yv + zw)^2}{u^2 + v^2 + w^2}$$

Without loss of generality, we assume density 1, so mass $= 8a^3$.

Now we have to calculate

$$
\begin{aligned}
I &:= \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} s^2\,dx\,dy\,dz \\
&= \frac{2}{3}a^3(2a)^2 + \frac{2}{3}a^3(2a)^2 + \frac{2}{3}a^3(2a)^2 - \frac{1}{u^2+v^2+w^2}\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}(xu+yv+zw)^2\,dx\,dy\,dz \\
&= 8a^5 - \frac{1}{u^2+v^2+w^2}\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}(x^2u^2+y^2v^2+z^2w^2+2uvxy+2uwxz+2vwyz)\,dx\,dy\,dz \\
&= 8a^5 - \frac{1}{u^2+v^2+w^2}\left((u^2+v^2+w^2)\frac{8}{3}a^5 + (uv+uw+vw)\cdot 0\right) \\
&= \frac{16}{3}a^5 = \frac{2}{3}(\text{mass})a^2
\end{aligned}
$$

Hwk #51:

(a) Calculate the area of the part of the graph $z = xy$ that is above the circle $x^2+y^2 \le a^2$. (That should lead to an easy integral.)

(b) Calculate the area of the part of the graph $z = xy$ that is above the square $|x| \le a$ and $|y| \le a$. (Setting up the integral is just as easy. But evaluating it requires some labor. Review Calc 2 integration skills or ask for substitution hints as needed.)

Solution: With x, y as coordinates on the surface, the parametrization is $\begin{bmatrix} x \\ y \\ xy \end{bmatrix}$. The area element is 5 pts

$$
d\text{area} = \left\| \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} \right\| dx\,dy = \sqrt{y^2+x^2+1^2}\,dx\,dy
$$

So we calculate the area in part (a), using polar coordinates with trivial angle integration, as:

$$
A_{(a)} = \int_{\text{disc}} \sqrt{1+x^2+y^2}\,d(x,y) = 2\pi\int_0^a \sqrt{1+r^2}\,r\,dr = \frac{2}{3}\pi\left[(1+r^2)^{3/2}\right]_0^a = \frac{2}{3}\pi\left((1+a^2)^{3/2}-1\right)
$$

For part (b), polar coordiantes are of no use, and we integrate in cartesian coordinates instead:

$$
A_{(b)} = \int_{\text{square}} \sqrt{1+x^2+y^2}\,d(x,y) = \int_{-a}^{a}\int_{-a}^{a}\sqrt{1+x^2+y^2}\,dx\,dy
$$

The best substitution to evaluate an integral of the form $\int\sqrt{b^2+x^2}\,dx$ is probably $x = b\sinh t$, even though the trig substitution $x = b\tan\phi$ also removes the square root. So

$$
\int\sqrt{b^2+x^2}\,dx = \int b^2\cosh^2 t\,dt = \frac{1}{2}b^2(t+\sinh t\cosh t) = \frac{1}{2}\Big(b^2\,\text{arsinh}\,\frac{x}{b} + x\sqrt{b^2+x^2}\Big)
$$

and we continue our evaluation

$$
A_{(b)} = 4\int_0^a\int_0^a\sqrt{1+x^2+y^2}\,dx\,dy = 2\int_0^a\left((1+y^2)\,\text{arsinh}\,\frac{a}{\sqrt{1+y^2}} + a\sqrt{1+y^2+a^2}\right)dy
$$

It is probably the best to get rid of the arsinh via an integration by parts of the first term:

$$\begin{aligned}\int (1+y^2)\operatorname{arsinh}\frac{a}{\sqrt{1+y^2}}\,dy &= \Big(y+\frac{1}{3}y^3\Big)\operatorname{arsinh}\frac{a}{\sqrt{1+y^2}} - \int \Big(y+\frac{1}{3}y^3\Big)\frac{-a(1+y^2)^{-3/2}y}{\sqrt{1+\frac{a^2}{1+y^2}}}\,dy\\ &= \Big(y+\frac{1}{3}y^3\Big)\operatorname{arsinh}\frac{a}{\sqrt{1+y^2}} + a\int \Big(y^2+\frac{1}{3}y^4\Big)\frac{(1+y^2)^{-1}}{\sqrt{1+y^2+a^2}}\,dy\end{aligned}$$

Now we continue the evaluation

$$A_{(b)} = \Big(2a+\frac{2}{3}a^3\Big)\operatorname{arsinh}\frac{a}{\sqrt{1+a^2}} + 2a\int_0^a \left[\Big(y+\frac{1}{3}y^3\Big)\frac{(1+y^2)^{-1}y}{\sqrt{1+y^2+a^2}} + \sqrt{1+y^2+a^2}\right] dy$$

The big bracket can be slightly simplified to $\dfrac{1+a^2+(3+a^2)y^2+\frac{4}{3}y^4}{(1+y^2)\sqrt{1+a^2+y^2}}$, and then the integral is treated with the substitution $y=\sqrt{1+a^2}\sinh s$. Temporarily introducing the abbreviation $\sqrt{1+a^2} =: c$, we get

$$A_{(b)} = \Big(2a+\frac{2}{3}a^3\Big)\operatorname{arsinh}\frac{a}{c} + 2a\int_0^{\operatorname{arsinh}(a/c)} \left[\frac{1+a^2+(3+a^2)c^2\sinh^2 s+\frac{4}{3}c^4\sinh^4 s}{(1+c^2\sinh^2 s)\,c\cosh s}\right] c\cosh s\,ds$$

A convenient substitution to transform the hyperbolic functions into rational functions is $\tanh s = u$, $\sinh^2 s = u^2/(1-u^2)$, $\cosh^2 s = 1/(1-u^2)$, $ds = du/(1-u^2)$. We'll have to figure out the new integration limit $u_* := \tanh\operatorname{arsinh}(a/c)$ later.

$$\begin{aligned}A_{(b)} &= \Big(2a+\frac{2}{3}a^3\Big)\operatorname{arsinh}\frac{a}{c} + 2a\int_0^{u_*}\left[\frac{1+a^2+(3+a^2)c^2u^2/(1-u^2)+\frac{4}{3}c^4u^4/(1-u^2)^2}{1+c^2u^2/(1-u^2)}\right]\frac{du}{1-u^2}\\ &= \Big(2a+\frac{2}{3}a^3\Big)\operatorname{arsinh}\frac{a}{c} + 2a\int_0^{u_*}\left[\frac{(1+a^2)(1-u^2)^2+(3+a^2)c^2u^2(1-u^2)+\frac{4}{3}c^4u^4}{1-u^2+c^2u^2}\right]\frac{du}{(1-u^2)^2}\end{aligned}$$

Now, we're in for a glorious partial fraction decomposition, and it is also time to rewrite the abbreviation c^2 as $1+a^2$ again. The integrand becomes

$$\begin{aligned}&\frac{(1+a^2)(1-u^2)^2+(3+a^2)(1+a^2)u^2(1-u^2)+\frac{4}{3}(1+a^2)^2u^4}{(1+a^2u^2)(1-u^2)^2} =\\ &= \frac{A}{1+a^2u^2}+\frac{B}{1-u^2}+\frac{C}{(1-u^2)^2}\\ &= \frac{A}{1+a^2u^2}+\Big(\frac{B}{2}+\frac{C}{4}\Big)\Big(\frac{1}{1-u}+\frac{1}{1+u}\Big)+\frac{C}{4}\Big(\frac{1}{(1-u)^2}+\frac{1}{(1+u)^2}\Big)\end{aligned}$$

Note that it is algebraically simpler to do the PFD first with u^2 rather than u as the variable. The cover-up method at $u^2=1$ gives $C=\frac{4}{3}(1+a^2)$. The cover-up method at $u^2=-1/a^2$ gives $A=-\frac{2}{3}$. The cover-up as $u^2\to\infty$ (i.e., multiplying by u^2 and taking the limit as $u^2\to\infty$) gives $A/a^2-B=\frac{1}{3}(a^2+1)(a^2-2)/a^2$ and therefore $B=\frac{1}{3}(1-a^2)$.

We also calculate $u_*=\tanh t_*$, where $t_*>0$ satisfies $\sinh t_*=\frac{a}{c}$. Therefore $\tanh t_* = \frac{a/c}{\sqrt{1+(a/c)^2}} = \frac{a}{\sqrt{a^2+c^2}}$. Remembering $0<u_*<1$, we integrate:

$$A_{(b)} = \Big(2a+\frac{2}{3}a^3\Big)\operatorname{arsinh}\frac{a}{c} + 2A\arctan au_* + a\Big(B+\frac{C}{2}\Big)\ln\frac{1+u_*}{1-u_*} + \frac{aC}{2}\Big(\frac{1}{1-u_*}-\frac{1}{1+u_*}\Big)$$

Using arsinh $\frac{a}{c} = \ln(\frac{a}{c} + \sqrt{1 + (\frac{a}{c})^2}) = \ln(a + \sqrt{a^2 + c^2}) - \ln c$, we can combine the arsinh and the ln term, in particular since $a(B + \frac{C}{2}) = a + \frac{1}{3}a^3$. So we get

$$A_{(b)} = \Big(a + \frac{1}{3}a^3\Big) \ln\left[\frac{(a + \sqrt{a^2 + c^2})^2(\sqrt{a^2 + c^2} + a)}{c^2(\sqrt{a^2 + c^2} - a)}\right] - \frac{4}{3}\arctan\frac{a^2}{\sqrt{a^2 + c^2}} + \frac{4}{3}a(1 + a^2)\Big(\frac{a/\sqrt{a^2 + c^2}}{1 - a^2/(a^2 + c^2)}\Big)$$

$$= 4\Big(a + \frac{1}{3}a^3\Big)\ln\frac{a + \sqrt{2a^2 + 1}}{\sqrt{1 + a^2}} - \frac{4}{3}\arctan\frac{a^2}{\sqrt{2a^2 + 1}} + \frac{4}{3}a^2\sqrt{2a^2 + 1}$$

After such a hefty calculation, one is glad to have, as a consistency check, some obviously true property of the area that this formula must fulfill.

For one thing, we expect $A_{(b)} \approx 4a^2$ for small a, because there the graph of $z = xy$ is approximated well by its tangent plane $z = 0$. A Taylor expansion up to order a^2 confirms this. (Remember to do this not by taking derivatives, but by algebra of elementary power series.) On the other hand, $A_{(b)}$ should lie between $\frac{2}{3}\pi((1 + a^2)^{3/2} - 1)$ and $\frac{2}{3}\pi((1 + 2a^2)^{3/2} - 1)$, because the square $[-a, a] \times [-a, a]$ contains the disc of radius a and is contained in the disc of radius $\sqrt{2}a$, which allows comparison of $A_{(b)}$ with the much easier $A_{(a)}$. We could plot the three formulas to confirm this numerically. For large a, our formula is approximately $A_{(b)} \approx \frac{4}{3}a^3(\ln(1 + \sqrt{2}) + \sqrt{2})$, which is at least between the approximate values $\frac{2\pi}{3}a^3$ and $\frac{4\pi}{3}\sqrt{2}\,a^3$ for $A_{(a)}$ for the corresponding discs. Odds are that random miscalculations may lead to a result in violation of one of these obvious inequalities. So these simple checks build some confidence in our calculational work.

The most powerful plausibility check (if you have good software) is to do a numerical integration and to compare it with the numerical evaluation of our messy formula for a few sample values of a.