Hwk #47:

Calculate the Jacobi matrix (total derivative) of the coordinate transformation

$$\begin{aligned} x &= x(r,\vartheta,\varphi) = r\sin\vartheta\,\cos\varphi \\ y &= y(r,\vartheta,\varphi) = r\sin\vartheta\,\sin\varphi \\ z &= z(r,\vartheta,\varphi) = r\cos\vartheta \end{aligned}$$

and obtain the volume element in these (spherical) coordinates. Also for fixed r obtain the surface area element.

Use these to calculate the volume of a ball of radius R, and its surface area, directly from the multivariable integral formalism. *Of course you know these formulas already and can also obtain them from single variable integral methods applied to rotation surfaces and bodies. But here the punchline is that you use the newly acquired methods on this simple test case.*

Hwk #48:

The curve $y = f(x)$ $(x_0 \le x \le x_1)$ rotates about the x axis. Obtain the formula from single variable calculus as a special case from the area formula for general surfaces.

Hwk #49:

Physicists are familiar with the following phenomenon: If you let a massive ball and a massive cylinder roll down an incline, then the ball rolls more rapidly than the cylinder. The reason is that part of the potential energy gained when losing height is converted into kinetic energy for the forward movment, whereas another part is converted into 'internal' (rotational) kinetic energy, because the object is rolling rather than just sliding. This rotational energy is lost to the forward motion.

You may know the formula $\frac{1}{2}mv^2$ (half mass times velocity^2) for the translation energy. There is a similar formula $\frac{1}{2}I\omega^2$ for the rotation energy, where ω measures how many radians per time unit an object rotates. The quantity I is called 'moment of inertia' and it depends on the mass distribution in the body. Mass that is closer to the rotation axis counts less because it does not move as fast as mass that is farther away from the rotation axis.

The formula for I is: $I = \int_{\text{body}} s^2 \rho d\text{vol}(x,y,z)$. Here ρ is the density (which may depend on (x,y,z), but in this problem we assume it is constant). s denotes the distance from the rotation axis, which you have to express in terms of x,y,z or whatever coordinates you use.

Given this wisdom, I ask you to find I for a cylinder of radius R and height h, and also for a ball of radius R. In either case, these objects rotate about a symmetry axis. You are to express the result in the form: number times (total mass) times R^2. Remember that the total mass is volume times density ρ.

The larger the number in front of 'mass times R^2' is, the higher the proportion of energy that is used for the rotation.

Hwk #50:

Now we rotate a cube $-a \le x, y, z \le a$ about an axis through the origin. The axis goes in the direction of a vector $\vec{v}$.

First draw a generic picture of a vector $\vec{x} = [x,\, y,\, z]^T$ and a vector $\vec{v} = [v_1,\, v_2,\, v_3]^T$ (both starting at the origin) and find a formula for the distance s of the tip of $\vec{x}$ (i.e., of the point (x, y, z)) from the axis that goes along the vector $\vec{v}$.

Then calculate the moment of inertia for this rotation (expressed as number times mass times a^2). Surprise: The final result will not depend on $\vec{v}$ — (To those who know about the tensor of inertia and the role eigenvalues play there, this surprise will be expected; but these wise folks, that's not us, for the time of Calc 3.)

Hwk #51:

(a) Calculate the area of the part of the graph $z = xy$ that is above the circle $x^2 + y^2 \le a^2$. (That should lead to an easy integral.)

(b) Calculate the area of the part of the graph $z = xy$ that is above the square $|x| \le a$ and $|y| \le a$. (Setting up the integral is just as easy. But evaluating it requires some labor. Review Calc 2 integration skills or ask for substitution hints as needed.)