Hwk #39:

This problem gives you the most celebrated use of Lagrange multipliers, but it requires some intrduction to appreciate it. (The calculations aren't bad at all.)

A famous task in linear algebra and matrix theory is to find *eigenvalues* of a given matrix. If A is a square matrix and you can find a *non-zero* vector v such that Av is actually a multiple of v, i.e., λv where λ is a number, then we call λ an eigenvalue of the matrix A (and v an eigenvector). For instance, the matrix $A = \begin{bmatrix} 3 & 2 \\ 3 & -4 \end{bmatrix}$ has 2 as an eigenvalue and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ as a corresponding eigenvector, because

$$\begin{bmatrix} 3 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

It also has -3 as an eigenvalue with $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ as an eigenvector. Of course multiples of eignevectors are again eigenvectors, e.g., if $Av = 2v$ then also $A(7v) = 2(7v)$. — In the example, there are only these two numbers $\lambda_1 = 2$ and $\lambda_2 = -3$ that are eigenvalues. If you try to find $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ solving $Av = \lambda v$ for any other λ you will only get the solution $v_1 = v_2 = 0$, i.e., only the zero vector. (Try it, just to gain familiarity with the notions.)

This problem is about eigenvalues of *symmetric* matrices. They play a role in studying definiteness of symmetric matrices. In physics, they are key concepts in describing rotating motions of rigid bodies. To every body, there is associated a symmetric 3×3 matrix called its 'tensor of inertia', whose eigenvectors point in the directions of such axes about which the body can rotate without wobbling (i.e., in a balanced way). The eigenvalues are called the moments of inertia about these axes.

To every symmetric $n \times n$ matrix A we associate the quadratic function $f(x) := x^T Ax$ where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$. We try to minimize or maximize $f(x)$ under the constraint $x^T x = 1$ (i.e., for x on the unit sphere).

(a) Write out $f(x)$ in components x_i for a 3×3 matrix A whose entries are called a_{ij}. Explain why a global maximum and a lobal minimum of $f(x)$ on the sphere are a-priori guaranteed to exist.

(b) Use the Lagrange multiplier method to set up equations satisfied by the x prooviding a minimum or a maximum. (You may have written all these in components; but now make sure to rewrite the whole stuff again in matrix and vector form.) While you are not asked to actually solve for x (that would be very tedious, involving a cubic equation for λ), I ask you to express the value of f at the minimum and maximum in terms of the Lagrange multiplier. [Be aware that when finding the max vs the min, x and λ will typically refer to numerically different quantities in these two cases.]

You have just proved that every symmetric 3×3 matrix has (at least) two real eigenvalues. (Actually, if A is a multiple of the identity matrix, these two eigenvalues coincide.) And with jut a bit more writing, the same can be done for symmetric $n \times n$ matrices.

The method can be cranked up a bit, by throwing in further constraints, to prove that every symmetric $n \times n$ matrix has n real eigenvalues (some of which may coincide). This may well be among the most important pieces of insight in undergraduate mathematics, and it's a pity that it often falls between the cracks of separating Calc 3 and Linear Algebra into independent courses of the curriculum.

(c) Show, in a very brief calculation: If A is positive definite, then all its eigenvalues are positive. If A is positve semidefinte, then all of its eigenvalues are ≥ 0. FYI: The converse is also true; so indeed a symmetric matrix is positve definite (resp. emidefinite) IF AND ONLY IF all of its eigenvalues are positive (resp. non-negative). This statement is sponsored by the above proof (a), (b) and some extra dose of linear algebra. It is the launch pad for proving the Hurwitz and Gershgorin tests I gave you before.

Hwk #40:

Think of the task of finding the absolute maximum of $x^2 + \frac{1}{2}y^2 + y^4 - xy$ on the set S given by $(x-1)^2 + |y + y^3| \leq 5$. The purpose of this problem is NOT that you would actually do calculations to FIND the maximum (which would require numerical methods). Rather, in preparation for such a search. I want you to use the Hessian to conclude that the maximum exists and is on the BOUNDARY of the set S.

The message here is: While modest problems can already lead to prohibitively complicated calculations that may need numerical tools, simple analytic arguments may still be able significantly to reduce the amount of labor in a numerical search.

Hwk #41:

A variation of the theme from the previous problem that is more 'real life', coming straight from partial differential equations. Suppose we have a bounded domain Ω in $\mathbb{R}^3$ and a function f that is continuous on the closure $\bar{\Omega}$ and C^2 in Ω. Suppose f satisfies $\partial^2 f(x,y,z)/\partial x^2 + \partial^2 f(x,y,z)/\partial y^2 + \partial^2 f(x,y,z)/\partial z^2 = 1$ in Ω, and $f = 0$ on the boundary. Show that then $f < 0$ in Ω. *Hint: write down the diagonal of the Hessian and show that necessary conditions for the negative definiteness are incompatible with the information given about f.* — Note: This kind of reasoning is known under the name of 'maximum principle'

Hwk #42:

Let T be the set $\{(x,y) \mid 0 \leq x \leq \pi,\ 0 \leq y \leq x\}$. Draw a figure of this set. Then evaluate the integral $I := \int_T \sin x \, \sin y \, d(x,y)$ in two ways: as iterated integral in either order.

Hint: Make sure you get the limits of integration right. If any of your calculations leaves a dangling x or y in the result you sure haven't gotten the limits right. This alert applies to all MV integral problems.

Hwk #43:

Let A be the set $\{(x,y) \mid 1 \leq x^2+y^2 \leq 4,\ x,y \geq 0\}$. Draw a figure of this set. Then evaluate the integral $I := \int_A x^2 y \, d(x,y)$ in two ways: one version using cartesian coordinates, and one using polar coordinates.

Using cartesian coordinates here is a bit dumb, admittedly. But I am asking that you do it anyways, to see the comparison with polar coordinates, and as a training to deal with the limits of integration correctly. Note that one order of integration in cartesian coordinates is easier to calculate than the other. Can you see which, and why?

Hwk #44:

Let $\vec{u} = [2, 1, 3]^T$, $\vec{v} = [-1, 0, 4]^T$, $\vec{w} = [2, -1, -3]^T$. Calculate $\vec{v} \times \vec{w}$, $\vec{u} \times (\vec{v} \times \vec{w})$, $\vec{u} \times \vec{v}$, $(\vec{u} \times \vec{v}) \times \vec{w}$.

With these same vectors from the previous problem, calculate $\vec{u} \cdot (\vec{v} \times \vec{w})$ and $\vec{w} \cdot (\vec{u} \times \vec{v})$.

Hwk #45:

Find the area of the triangle whose vertices are the points $A(1, 1, 3)$, $B(-2, 3, 0)$, $C(1, 1, -2)$.

Hwk #46:

Given the vectors $\vec{u} = [u_1, u_2, u_3]^T$ and $\vec{v} = [v_1, v_2, v_3]^T$ in space, I have defined their cross product $\vec{u} \times \vec{v}$ to be the vector $\vec{w} = [u_2 v_3 - u_3 v_2\,,\ u_3 v_1 - u_1 v_3\,,\ u_1 v_2 - u_2 v_1]^T$. Show that indeed, $\|\vec{w}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2 \varphi)$, where φ is the angle between $\vec{u}$ and $\vec{v}$.