Hwk #32:

This example is taken from *I. Rosenholtz, L. Smylie: "The only Critical Point in Town" Test*, Mathematics Magazine **58**(1985), 149–150.

Show that the function

$$g : (x, y) \mapsto y^2 + 3(y + e^x - 1)^2 + 2(y + e^x - 1)^3 \,,\ \mathbb{R}^2 \to \mathbb{R}$$

has exactly one critical point, and that this point is a relative mimimum.

Furthermore explain why this point is NOT an absolute minimum.

Solution: It is convenient to use the abbreviation $u := y + e^x - 1$. 5 pts

$$\frac{\partial g(x,y)}{\partial x} = 6ue^x + 6u^2e^x \qquad \frac{\partial g(x,y)}{\partial y} = 2y + 6u + 6u^2$$

For both to vanish we need $y = 0$ and $u + u^2 = 0$, which latter means $u = 0$ or $u = -1$. Since $y = 0$, $u = e^x - 1$, and this cannot equal -1. So $u = 0$, and this means $x = 0$.

Therefore the only critical point is $(x, y) = (0, 0)$. Let's calculate the Hessian:

$$Hg(x,y) = \begin{bmatrix} (6u + 6u^2)e^x + (6 + 12u)e^{2x} & (6 + 12u)e^x \\ (6 + 12u)e^x & 8 + 12u \end{bmatrix}$$

$$Hg(0,0) = \begin{bmatrix} 6 & 6 \\ 6 & 8 \end{bmatrix}$$

This matrix is positive definite, by the Hurwitz test: $h_{11} = 6 > 0$ and $h_{11}h_{22} - h_{12}^2 = 12 > 0$. Therefore the origin is a relative minimum of g. $g(0, 0) = 0$. But $g(0, y) = 4y^2 + 2y^3 \to -\infty$ as $y \to -\infty$. The function is unbounded below and does not have an absolute minimum.

Comment: This example shows two things. Firstly, it does not suffice to check the values of a function at the only candidates for a relative minimum in order to determine a global minimum, *unless* the existence of an absolute minimum is established beforehand.

But there is a second lesson hidden in this example. An intuitively plausible argument would go like this: "If I am hiking in a landskape and standing in a local minimum, but there are points out there with lower elevation than my present location, then it must be possible, in principle, to reach them by a walk that goes through some pass (mathematically a saddle point). So, if a local minimum is not a global minimum, there should be another critical point, a saddle point, somewhere." This argument does actually have some merit. However, the saddle point could have run away to infinity. In our example the 2-variable function $(u, y) \mapsto y^2 + 3u^2 + 2u^3$ has a local minimum at $(u, y) = (0, 0)$ and a saddle point at $(u, y) = (-1, 0)$. But $u \to -1$ corresponds to $x \to -\infty$ when $u = y + e^x - 1$ and $y = 0$.

In advanced applications, this principle 'there should be a saddle point, if only we can make sure that it hasn't run off to infinity' is a powerful tool in solving partial differential equations.

Hwk #33:

This example is taken from Marsden-Tromba: Show that the function f given by $f(x, y) = (y - 3x^2)(y - x^2)$ has a critical point in the origin, which is neither a relative minimum nor a relative maximum. What kind of '***'definite is the Hessian?

Show also that all single-variable radial functions $t \mapsto f(t\cos\phi, t\sin\phi)$ have a relative minimum at $t = 0$.

Solution: For fixed ϕ, let $g(t) := f(t\cos\phi, t\sin\phi) = t^2\sin^2\phi - 4t^3\sin\phi\cos^2\phi + 3t^4\cos^4\phi$. Then $g'(0) = 0$ and $g''(0) = 2\sin^2\phi$. For $\sin\phi \neq 0$, we can argue that g has a local minimum at 0 because the first derivative vanishes and the second derivative is positive there. For $\sin\phi = 0$, we have $\cos^2\phi = 1$ and $g(t) = 3t^4$, and we again have a local minimum at 0, albeit a 'degenerate' one that cannot be detected by the second derivative test. . 5 pts

Now clearly $f(0,0) = 0$, but there are both positive and negative values in any neighbourhood of $(0,0)$. For instance $f(x, 2x^2) = -x^4 < 0$. Let's calculate the Hessian from $f(x,y) = 3x^4 - 4x^2y + y^2$: $f_{xx}(x,y) = 36x^2 - 8y$, $f_{xy}(x,y) = -8x$, $f_{yy}(x,y) = 2$. So $Hf(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. This matrix is positive semidefinite, but not positive definite.

Note: Similar as with continuity and differentiability, we learn here for the local minimu property that the MV version cannot be captured by having the single variable version in all directions. However, a slightly strengthened version of the single variable local minimality (namely that the sufficient condition of having positive second derivative is satisfied) in all directions does suffice to prove local minimality in the multi-variable context for C^2 functions.

Hwk #34:

This example is geometrically appealing, but alas calculationally lengthy. This is why I give you the intermediate steps and hints to navigate you through. Ideally, it should be done with the help of symbolic algebra software, and you are welcome to use this tool, if available.

We want to find a shortest connection between two plane curves, namely $y = x^2 + 2$ and $y = \frac{1}{2}(x-1)^2$. A precise plot is attached. Choose points $P = (a, a^2+2)$ on the first parabola and $Q = (b, \frac{1}{2}(b-1)^2)$ on the second and minimize the square of the distance. Determine all critical points and classify them. Does one of them probide a *global* minimum? Why?

Hint 1: While it is possible to take one of the equations for a critical point and solve it for b via by means of the quadratic formula, and then plug in the result in the other equation, this is tedious. It is more straightforward to take successively linear combinations of the two equations with the strategy of first eliminating b^3, then b^2, then b, until one polynomial equation in a remains.

Hint 2: After an obvious factorization of this polynomial equation, an easy solution $a = 1$ can be guessed, and when this is factored off, a 4th order polynomial remains that can be factored into two quadratics with integer coefficients; indeed one factor is $a^2 + 2a + 3$.

Solution: We take the distance squared between a point $P = (a, a^2+2)$ on the first parabola and a point $Q = (b, \frac{1}{2}(b-1)^2)$ on the second parabola and get the function 6 pts

$$f(a,b) := (a-b)^2 + \Big(a^2 + 2 - \frac{1}{2}(b-1)^2\Big)^2 .$$

We obtain

$$\partial f(a,b)/\partial a = 2(a-b) + 4a\Big(a^2 + 2 - \frac{1}{2}(b-1)^2\Big) = 0$$

$$\partial f(a,b)/\partial b = 2(b-a) - 2(b-1)\Big(a^2 + 2 - \frac{1}{2}(b-1)^2\Big) = 0$$

So we have to solve the system of two polynomial equations

$$-b + 2a^3 + 4a - ab^2 + 2ab = 0 \quad (1)$$
$$2a^2 - 2a + 3 - 2a^2b + b^3 - 3b^2 + b = 0 \quad (2)$$

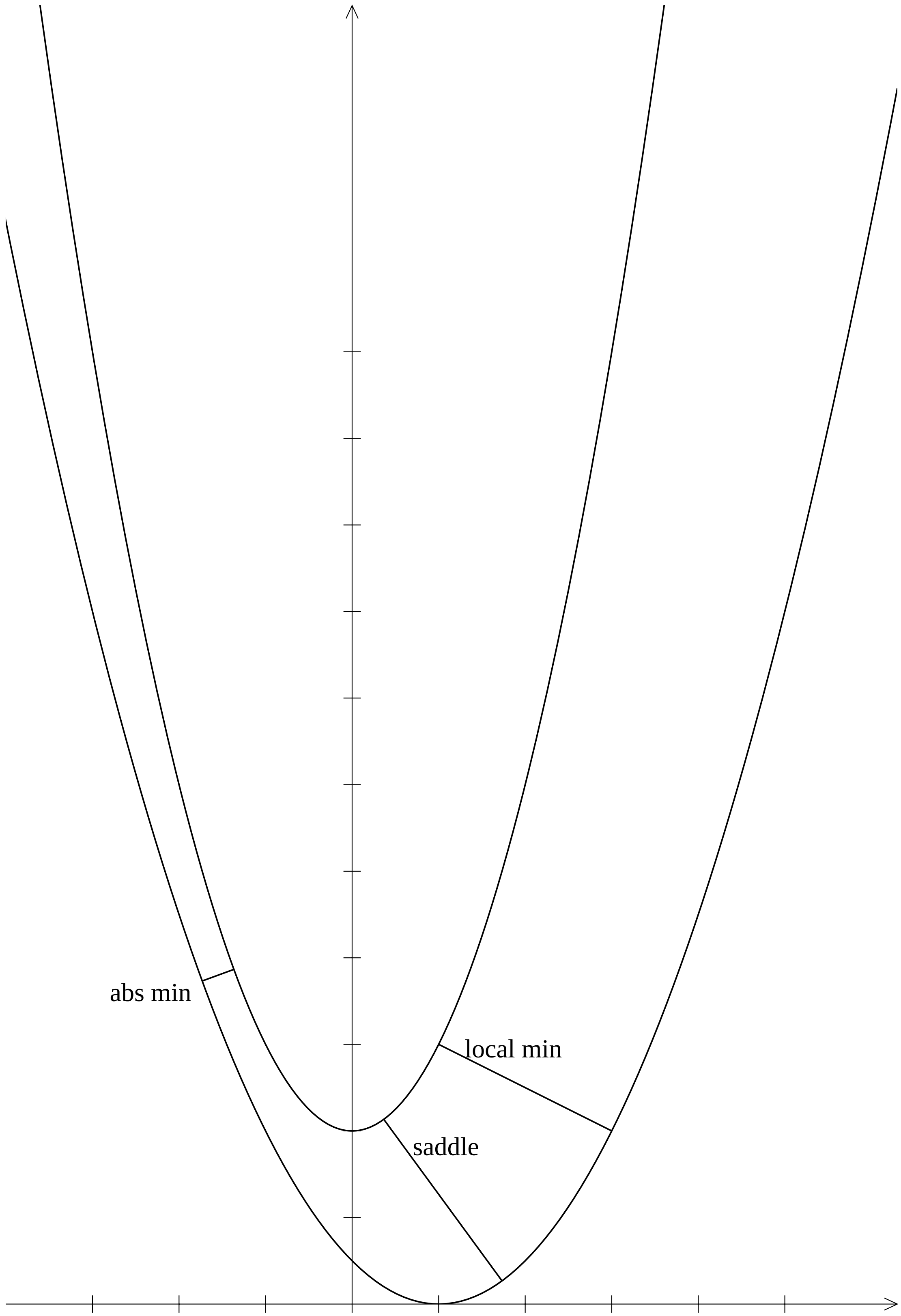


Figure 1: Figure for Hwk # 34

Following the hint, we take a times (2) plus b times (1) to get rid of b^3. We obtain

$$2a^3 - 2a^2 + 3a + 5ab - (a+1)b^2 = 0 \quad (3)$$

Now we add $-(a+1)$ times (1) to a times (3) to eliminate b^2. We obtain

$$-4a^3 - a^2 - 4a + b(3a^2 - a + 1) = 0 \quad (4)$$

We solve this for b and plug it back into (1):

$$2a^3 + 4a + (2a-1)\frac{4a^3 + a^2 + 4a}{3a^2 - a + 1} - a\left(\frac{4a^3 + a^2 + 4a}{3a^2 - a + 1}\right)^2 = 0$$

Clearing denominators and expanding, we get

$$2a^7 + 4a^6 + 3a^5 - 5a^4 - 7a^3 + 3a^2 = 0 \quad (5)$$

We can factor off $a = 0$ twice and guess the solution $a = 1$, thus factoring off $(a-1)$. After long division of polynomials, we get

$$0 = 2a^4 + 6a^3 + 9a^2 + 4a - 3 = (2a^2 + 2a - 1)(a^2 + 2a + 3)$$

This factorization, short of the hint given, would best be found by means of a symbolic algebra software. $a^2 + 2a + 3 = 0$ doesn't have real solutions; $2a^2 + 2a - 1 = 0$ can be solved by the quadratic formula. We have thus found the following solutions of (5):

$$a_0 = 0\,, \quad a_1 = 1\,, \quad a_{2,3} = \frac{1}{2}\left(-1 \pm \sqrt{3}\right)$$

We can get b form a by means of (4), but initially, all of these are mere *consequences* of (1),(2); they are not *equivalent* to (1),(2). We must plug them back into the original equations. Clearly $a_0 = 0$ violates (2). For $a \neq 0$, (1),(2) $\iff$ (1),(3) $\iff$ (1),(4) $\iff$ (4),(5). The last equivalence uses $3a^2 - a + 1 \neq 0$. So we have found

$$\begin{array}{lll}
(a_1, b_1) = (1, 3) & P_1 = (1, 3) & Q_1 = (3, 2) \\
(a_2, b_2) = \left(\frac{1}{2}\left(-1+\sqrt{3}\right), \sqrt{3}\right) & P_2 = \left(\frac{1}{2}\left(-1+\sqrt{3}\right), 3 - \frac{1}{2}\sqrt{3}\right) & Q_2 = \left(-\sqrt{3}, 2 - \sqrt{3}\right) \\
(a_3, b_3) = \left(\frac{1}{2}\left(-1-\sqrt{3}\right), -\sqrt{3}\right) & P_3 = \left(\frac{1}{2}\left(-1-\sqrt{3}\right), 3 + \frac{1}{2}\sqrt{3}\right) & Q_3 = \left(\sqrt{3}, 2 + \sqrt{3}\right)
\end{array}$$

Numerical values are:

$$\begin{array}{ll|l}
P_1 = (1, 3) & Q_1 = (3, 2) & |P_1Q_1| = 2.236 \\
P_2 = (0.366, 2.134) & Q_2 = (1.732, 0.268) & |P_2Q_2| = 2.313 \\
P_3 = (-1.366, 3.866) & Q_3 = (-1.732, 3.732) & |P_3Q_3| = 0.390
\end{array}$$

The Hessian of f is

$$Hf(a,b) = \begin{bmatrix} 8 + 12a^2 + 4b - 2b^2 & -2 + 4a - 4ab \\ -2 + 4a - 4ab & 1 - 2a^2 - 6b + 3b^2 \end{bmatrix}$$

$$Hf(a_1,b_1) = \begin{bmatrix} 14 & -10 \\ -10 & 8 \end{bmatrix} \qquad \text{pos def}$$

$$Hf(a_2,b_2) = \begin{bmatrix} 14-2\sqrt{3} & -10+4\sqrt{3} \\ -10+4\sqrt{3} & 8-5\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 10.536 & -3.072 \\ -3.072 & -0.660 \end{bmatrix} \qquad \text{indefinite}$$

$$Hf(a_3,b_3) = \begin{bmatrix} 14+2\sqrt{3} & -10-4\sqrt{3} \\ -10-4\sqrt{3} & 8+5\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 17.464 & -16.928 \\ -16.928 & 16.660 \end{bmatrix} \qquad \text{pos def}$$

So (a_2, b_2) is a saddle point, the other two are local minima. The one with the smaller distance, namely (a_3, b_3) can be accepted as a global minimum, PROVIDED we know apriori that a global minimum EXISTS.

We cannot argue directly, because $\mathbb{R} \times \mathbb{R} \ni (a, b)$ is not bounded. However, we have the additional feature that $f(a, b) \to \infty$ as either a or b goes to infinity. For instance, when asking whether a global minimum exists, we can a-priori neglect all (a, b) for which $f(a, b) > 100$, because at some points f has smaller values, e.g., $f(0, 1) = 5 \ll 100$. So we only need to consider those (a, b) where $|a - b| \le 10$, because otherwise $f(a, b) > 10^2 + 0$. But if $|a - b| \le 10$, we have $a^2 + 2 - \frac{1}{2}(b-1)^2 = \frac{1}{2}a^2 + \frac{1}{2}[a^2 - (b-1)^2] + 2 \ge \frac{1}{2}a^2 + 2 - \frac{1}{2}|a-b+1|\,|a+b-1| \ge \frac{1}{2}a^2 + 2 - \frac{11}{2}(2|a| + 11)$. For $|a|$ sufficiently large, this will again be > 10, making $f(a, b) > 0 + 10^2$. So we need to consider only the absolute minimum on some closed and bounded set $\{(a, b) : |a - b| \le 10\,,\ |a| \le C\}$, and an abolute minimum exists on this set. Outside ths set, the values of f are larger, so we do have an absolute minimum on $\mathbb{R}^2$.

Hwk #35:

Suppose in the following matrices, the starred entries are not known. Which of the five possibilities 'positive definite', 'positive semidefinite (but not definite)', 'negative definite', 'negative semidefinite (but not definite)', 'indefinite' remains a possibility, based on knowledge only of the known entries?

$$(a)\begin{bmatrix} 3 & * \\ * & * \end{bmatrix} \qquad (b)\begin{bmatrix} * & * \\ * & -5 \end{bmatrix} \qquad (c)\begin{bmatrix} * & 6 \\ 6 & * \end{bmatrix} \qquad (d)\begin{bmatrix} 3 & * \\ * & -1 \end{bmatrix} \qquad (e)\begin{bmatrix} 0 & 1 & * \\ 1 & * & * \\ * & * & * \end{bmatrix}$$

Consider the following: While the Hurwitz test was worded in a way to calculate determinants starting from the top, the order in which the variables are listed (and thus determine entries of the matrix) is not essential for definiteness of a matrix; so you could use the determinants in the Hurwitz test starting at any diagonal element and then calculating 2×2, 3×3, etc. determinants, adding any one variable (row and column) at a time.

Solution: (a) this matrix could be positive (semi-)definite or indefinite, but not negative (semi-) definite, because the vector $[1\,,\ 0]^T$ makes the quadratic form positive. 5 pts

(b) this matrix could be negative (semi-)definite or indefinite, but not positive (semi-)definite, because the vector $[0\,,\ -1]^T$ makes the quadratic form negative.

(c) no conclusion can be drawn: If one diagonal element is positive and one negative, then the matrix is indefinite. If both diagonal elements are larger than 6, the matrix is positive definite by Gershgorin's test. If both diagonal elements are below -6, the matrix is negative definite. If both diagonal elements are equal 6 (or equal -6), the matrix is positive (or negative) semidefinite by explicit writing down of the quadratic form. (This does not exhaust all possibilities that the $*$ could stand for, but it already represents all cases for definiteness properties of the matrix.)

(d) this matrix is indefinite, regardless of the off-diagonal elements.

(e) this matrix is indefinite. The determinant of the 2×2 submatrix $\begin{bmatrix} 0 & 1 \\ 1 & * \end{bmatrix}$ is -1, so the matrix cannot be positive definite (and the 0 in the corner also confirms this). Similarly, the negative of this matrix cannot be positive definite either. – We do not have a Hurwitz test for semi-definiteness, so we cannot rule this case out by determinants. But let's just calculate the quadratic form with the vector $[1\,,\ t\,,\ 0]^T$. We get $0 \cdot 1^2 + 2 \cdot 1 \cdot t + * \cdot t^2 = 2t + *t^2$. Whatever the value of $*$, this expression is positive for t positive and sufficiently small, but is negative for t negative of sufficiently small absolute value.

Hwk #36:

Find the absolute minimum and absolute maximum of $x^2 + (y-1)^2 + z^2 - xyz$ on the ball $x^2+y^2+z^2 \le 3^2$. *Hint: For the boundary consideration, use the xz plane as equator plane for the spherical coordinates, to benefit from the symmetry of the problem. Otherwise formulas get obnoxiously messy.*

Solution: Since this is a continuous function on a bounded and closed set, we know that an absolute minimum and maximum *exist*. We do not need to calculate Hessians to check for local minima or maxima, because they are not asked. The absolute extrema can be found by selecting from critical points in the interior and critical points on the boundary. 5 pts

Let's first study any interior critical points of the function f given by $f(x,y,z) := x^2+(y-1)^2+z^2-xyz$: For the gradient to vanish, we need

$$\begin{aligned}
(\partial_1 f)(x,y,z) &= 2x - yz = 0 \\
(\partial_2 f)(x,y,z) &= 2(y-1) - xz = 0 \\
(\partial_3 f)(x,y,z) &= 2z - xy = 0
\end{aligned}$$

Combining the first and third condition, we get $2x^2 = 2z^2 = xyz$. So we conclude either (1) $x = z$, $y = 2$ or (2) $x = -z$, $y = -2$, or (3) $x = z = 0$. Plugging each case into the 2nd equation we get: two solutions $(x,y,z) = (\pm\sqrt{2}, 2, \pm\sqrt{2})$ from (1). Two solutions $(x,y,z) = (\pm\sqrt{6}, -2, \mp\sqrt{6})$ from (2). One solution $(x,y,z) = (0,1,0)$ from (3). Since we have only been drawing conclusions from this system, we need to check that all these five solutions do satisfy all three equations. They do. The solutions from (2) are not in the ball and may therefore be discarded.

We note $f(\pm\sqrt{2}, 2, \pm\sqrt{2}) = 1$ and $f(0,1,0) = 0$.

We parametrize the boundary by spherical coordinates: $y = 3\cos\vartheta$, $x = 3\sin\vartheta\,\cos\varphi$, $z = 3\sin\vartheta\,\sin\varphi$. While this is originally meant for $\varphi \in [0, 2\pi]$ and $\vartheta \in [0,\pi]$, we can extend the coordinates by 'wrapping over' to $\varphi, \vartheta \in \mathbb{R}$. This reflects the fact that the sphere does not have a boundary. We only need to look for interior critical points of the 2-variable function

$$\begin{aligned}
g(\vartheta,\varphi) &:= f(3\sin\vartheta\,\cos\varphi, 3\cos\vartheta, 3\sin\vartheta\,\sin\varphi) \\
&= 9\sin^2\vartheta + (3\cos\vartheta - 1)^2 - 27\cos\vartheta\,\sin^2\vartheta\,\sin\varphi\,\cos\varphi \\
&= 10 - 6\cos\vartheta - \tfrac{27}{2}\cos\vartheta\,\sin^2\vartheta\,\sin 2\varphi
\end{aligned}$$

The vanishing of g_φ requires $\cos\vartheta\,\sin^2\vartheta\,\cos(2\varphi) = 0$. This means $\vartheta \in \{0, \frac{\pi}{2}, \pi\}$ or $\varphi \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$.

The vanishing of g_ϑ requires $6\sin\vartheta + \frac{27}{2}\sin^3\vartheta\,\sin 2\varphi - 27\sin\vartheta\,\cos^2\vartheta\,\sin 2\varphi = 0$.

$\vartheta \in \{0,\pi\}$ satisfy this condition and give rise to boundary critical points $f(0,3,0) = 4$ and $f(0,-3,0) = 16$.

$\vartheta = \frac{\pi}{2}$ (and hence $y = 0$) still needs $\sin 2\varphi = -\frac{4}{9}$ for g_ϑ to vanish. We do not need to pursue these critical points further because $g(\frac{\pi}{2}, \varphi) \equiv 10$, so they would be neither absolute minima nor absolute maxima. *Note: Lest this seem confusing: the function g is constant on the 'equator' $y = 0$, $x^2 + z^2 = 9$. But not all points on the equator are critical points, because criticality also depends on how the function changes off the equator.*

$\varphi \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$ still needs $6\sin\vartheta + \frac{27}{2}\sin^3\vartheta - 27\sin\vartheta\cos^2\vartheta = 0$ for g_ϑ to vanish. Apart from retrieving $\vartheta \in \{0, \pi\}$, which has been discussed already, we have to solve $6 + \frac{27}{2}(1 - \cos^2\vartheta) - 27\cos^2\vartheta = 0$, or $\cos\vartheta = \pm(\frac{13}{27})^{1/2}$. Then $\sin^2\vartheta = \frac{14}{27}$ and $g = 10 \mp 13(\frac{13}{27})^{1/2}$.

Similarly, $\varphi \in \{\frac{3\pi}{4}, \frac{7\pi}{4}\}$ still needs $6\sin\vartheta - \frac{27}{2}\sin^3\vartheta + 27\sin\vartheta\cos^2\vartheta = 0$ for g_ϑ to vanish. Apart from retrieving old critical points, we have to solve $6 - \frac{27}{2}(1 - \cos^2\vartheta) + 27\cos^2\vartheta = 0$, or $\cos\vartheta = \pm(\frac{5}{27})^{1/2}$. Then $\sin^2\vartheta = \frac{22}{27}$ and $g = 10 \pm 5(\frac{5}{27})^{1/2}$, values that do not qualify for a global extremum.

Concludingly, we have found the global minimum 0 at $(x, y, z) = (0, 1, 0)$ and the global maximum $10 + 13(\frac{13}{27})^{1/2} \approx 19.0206$ on the boundary at $(x, y, z) = (\pm(\frac{7}{3})^{1/2}, -(\frac{13}{3})^{1/2}, \pm(\frac{7}{3})^{1/2})$.

Hwk #37:

> Redo the boundary part of the calculations from the previous problem using Lagrange multipliers.

Solution: Minimizing/maximizing the expression $f(x, y, z) := x^2 + (y-1)^2 + z^2 - xyz$ on the level set $h(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$ leads to the following necessary conditions for a local extremum: 5 pts

$$\begin{aligned} 2x - yz &= 2\lambda x \quad (1)\\ 2(y-1) - xz &= 2\lambda y \quad (2)\\ 2z - xy &= 2\lambda z \quad (3)\\ x^2 + y^2 + z^2 &= 9 \quad (4) \end{aligned}$$

Guided by the symmetry between x and z, we seek to simplify this system by calculating $x\cdot(1) - z\cdot(3)$. After slight rearrangement, this gives $2(1-\lambda)(x^2 - z^2) = 0$. So either $x = \pm z$ or $\lambda = 1$. We also subtract $(1) - (3)$ directly and get $(2 + y)(x - z) = 2\lambda(x - z)$. So if $x \neq z$, we need $y = 2\lambda - 2$.

The case $\lambda = 1$ leads to $yz = 0 = xy$, $xz = -2$. So since x, z cannot vanish, we need $y = 0$. The combination $xz = -2$, $x^2 + z^2 = 9$ gives four solutions (the ones on the equator in the previous problem), and for all of them, the value of f is 10. We still have to consider the cases $x = z$ and $x = -z$.

Let's now look at the case $x = z$. Either $x = z = 0$ and hence $y = \pm 3$ with values 4 or 16 for f, or else we can cancel x from (1) and get $2\lambda = 2 - y$. Plugging this into (2) we get $2(y-1) - x^2 = (2 - y)y$. And (4) becomes $2x^2 + y^2 = 9$. These togehter are equivalent to $x^2 = \frac{7}{3}$, $y^2 = \frac{13}{3}$.

In the case $x = -z$ we may assume $x \neq z$ (hence $x, z \neq 0$), b/c $x = z$ has been studied already. So we have then $2\lambda = y + 2$ and this case leads to $-2 + x^2 = y^2$ from (2) and $2x^2 + y^2 = 9$ from (4). Hence $x^2 = \frac{11}{3}$ and $y^2 = \frac{5}{3}$. The values of f corresponding to this case are $10 \pm \frac{5}{3}(\frac{5}{3})^{1/2}$.

Hwk #38:

> (a) Given two points F_1 and F_2 in the plane and a curve $f(x, y) = 0$. Let the point P be restricted to this curve and assume P is neither F_1 nor F_2. Show: If P is such that the expression $\|\vec{F_1P}\| + \|\vec{F_2P}\|$ has a local maximum at P, then the path F_1PF_2 satisfies

the reflection property (as defined in the problem about the ellipse). What happens in the case of a local minimum? *Note that this time the curve is arbitrary, but the point on the curve is special.*

Billiard is the French name for the game known as 'pool' in the US. It is also the name for the mathematician's version of pool, regardless of country. Mathematical billiard is played on a pool table of arbitrary shape (rectangles are too boring), and it is played with a single mass point instead of balls (no spin or 'effet' can be given to a point, and it is immune to friction). It only bounces off the border, but not off any other mass points. Let's assume a *strictly convex* table with smooth boundary. Strictly convex means that the straight segment connecting any two points of the boundary passes through the interior of the table. Smooth boundary means that the boundary is the level line of a smooth function with nonvanishing gradient on that level line.

(b) I claim: Given any smooth strictly convex billiard and any natural number $n \geq 2$, there is a closed n-gon billiard path. Explain why.

So far you would have been thinking we have to solve equations in order to solve minimax problems. But in this example, you argue that a minimax problem must have a solution in order to prove that some complicated equations have a solution!

Solution: We are trying to maximize, or minimize, an expression $\|\vec{x}-\vec{a}\| + \|\vec{x}-\vec{b}\|$ subject to the constraint $f(\vec{x}) = 0$. Here $\vec{a}$ and $\vec{b}$ are position vectors of the points F_1 and F_2 respectively. [Position vector of a point: vector from the origin to a that point] Lagrange multipliers give the necessary condition 5 pts

$$\frac{\vec{x}-\vec{a}}{\|\vec{x}-\vec{a}\|} + \frac{\vec{x}-\vec{b}}{\|\vec{x}-\vec{b}\|} = \lambda \nabla f(\vec{x})$$

Multiplying this (by dot product) with $\nabla f(\vec{x})/\|\nabla f(\vec{x})\|$ seems enticing, but doesn't help, because we do not know the Lagrange multiplier λ. There are two ways to proceed Let $\vec{t}$ be a vector tangential to the level line (and hence orthogonal to the gradient). Then, dot-multiplying the Lagrange condition with $\vec{t}$ gives

$$\frac{(\vec{x}-\vec{a})\cdot\vec{t}}{\|\vec{x}-\vec{a}\|} - \frac{(\vec{x}-\vec{b})\cdot(-\vec{t})}{\|\vec{x}-\vec{b}\|} = 0$$

or the reflection condition in the form $\angle(\vec{x}-\vec{a},\vec{t}) = \angle(\vec{x}-\vec{b},-\vec{t})$. But hold on, there is an exceptional geometric situation where this is *not* a reflection condition; see figures after discussion of the alternate approach.

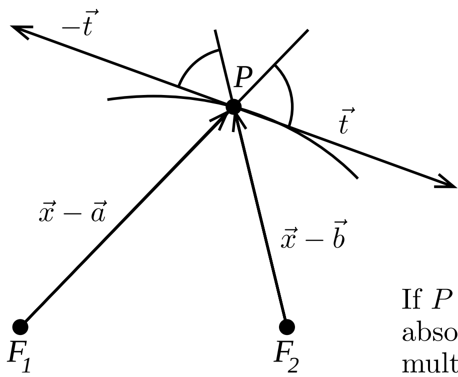
Alternatively, we can take turns dot-multiplying the Lagrange condition with $\frac{\vec{x}-\vec{a}}{\|\vec{x}-\vec{a}\|}$ and $\frac{\vec{x}-\vec{b}}{\|\vec{x}-\vec{b}\|}$ respectively. Either choice gives the same left hand side:

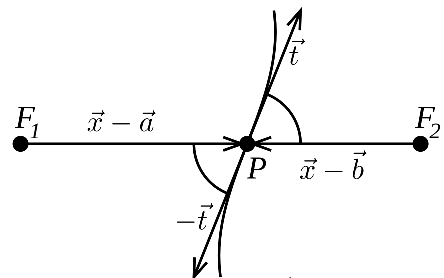
$$1 + \frac{(\vec{x}-\vec{a})\cdot(\vec{x}-\vec{b})}{\|\vec{x}-\vec{a}\|\,\|\vec{x}-\vec{b}\|} = \lambda\frac{\nabla f(\vec{x})\cdot(\vec{x}-\vec{a})}{\|\vec{x}-\vec{a}\|}$$
$$1 + \frac{(\vec{x}-\vec{a})\cdot(\vec{x}-\vec{b})}{\|\vec{x}-\vec{a}\|\,\|\vec{x}-\vec{b}\|} = \lambda\frac{\nabla f(\vec{x})\cdot(\vec{x}-\vec{b})}{\|\vec{x}-\vec{b}\|}$$

Therefore the right hand sides must also coincide, and we conclude that $\angle(\vec{x}-\vec{a},\nabla f(\vec{x})) = \angle(\vec{x}-\vec{b},\nabla f(\vec{x}))$ *unless* $\lambda = 0$. However, if $\lambda = 0$, then the Lagrange condition implies that $\vec{x}-\vec{a}$ and $\vec{x}-\vec{b}$ have opposite directions, i.e., the point P lies directly between F_1 and F_2. This situation minimizes the distance sum, but does not maximize it. Tha is, unless the level line of f is on the segment connecting

F_1 and F_2, in which case the distance sum is constant and therefore every point is trivially both a local maximum and a local minimum. But in this case the reflection property is trivially fulfilled, too.

If we assume that the level curve of f does not intersect the closed line segment from F_1 to F_2, then the case $\lambda = 0$ and P between F_1 and F_2 is ruled out automatically, and the same reasoning applies for a local minimum as does for a local maximum.





If P is between F_1 and F_2, we have an (unconstrained and absolute) minimum, which is also retrieved by the Lagrange multiplier method (with $\lambda = 0$, because it is an unconstrained minimum, so the gradient must vanish without including the constraint).

Now, to get a polygonal billard path, with corners P_1, P_2,. . . ,P_n, we consider a function ℓ of $2n$ variables giving the length of the polygonal path $P_1P_2 \ldots P_nP_1$. We set up n constraints, namely that each of the points P_i must be on the level line $f(x, y) = 0$. The wording of the problem needs to made a bit more precise. We have tacitly assumed, but should point out explicitly, that the pool table is bounded and that its boundary is therefore a closed curve. So what we are doing here will not apply, e.g., to billiard played 'inside a parabola'.

With this clarification, we are trying to maximize $\ell(P_1, \ldots P_n)$ (a continuous function) on a bounded and closed set, defined by the constraints $f(P_i) = 0$, and we are assured of the existence of an absolute maximum. (There may well be relative maxima, too.) We are also assured of the existence of an absolute minimum, but this minimum is not interesting: it occurs when all P_i coalesce and ℓ becomes 0.

We claim that the absolute maximum (and any further relative maximum, if one happens to exist; and any relative minimum with distinct points P_i, in case such a thing hould happen to exist also) gives a polygonal billard path. To this end, we argue that for a relative maximum, the P_i are automatically distinct (in a sense to be made more precise in a moment); this includes the case of an absolute maximum. (For a relative minimum, we need to assume that the P_i are distinct.) Once this is established, we notice that such a maximum remains a maximum if we freeze all but one points and consider ℓ as a function of this single point P_i only. This consideration proves the reflection property at P_i.

The P_i are distinct in the sense that $P_i \neq P_{i+1}$ and $P_n \neq P_1$, i.e., adjacent points on the polygonal path are distinct. (We define $P_{n+1} := P_1$ then we don't need to make such a distinction any more.) Assume P_i were equal to P_{i+1}. then the path $P_iP_{i+1}P_{i+2}$ would actually be just P_iP_{i+2}, and would be shorter than any other path $P_iP'_{i+1}P_{i+2}$ where we move P_{i+1} away from the P_i position along the curve $f(P_{i+1}) = 0$.

The polygon we obtain from our method may well have self-intersections, and if n is not a prime number, it may be a 'smaller' polygon traveled through multiple times over. For instance if the pool table is a ball and $n = 2$, we get the billiard point bouncing back and forth along a diameter. If $n = 3$, we get an equilateral triangle, but if $n = 4$ we do not get a square, but a diameter 'twice': P_1, P_2, $P_3 = P_1$, $P_4 = P_2$. For $n = 5$, you'd get a five-pronged star. For $n = 6$, the triangle again, run through twice. This looks like we wouldn't get quite what we wanted. However, there is another variant.

We could maximize ℓ under the constraints that the P_i are on the curve *and* are in cyclical order on the curve. These are inequality constraints; on the circle, they could be written as $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \ldots \leq \varphi_n \leq \varphi_1 + 2\pi$. We still have a closed and bounded set, so we still get a global maximum. But this time we need to consider the possibility of boundary cases ($\varphi_i = \varphi_{i+1}$), in which the Lagrang multiplier method would not apply. But our previous consideration that adjacent points in a maximum-length configuration must be distinct tells us that such a maximum is not taken on at the boundary. This method yields closed non-self-intersecting polygonal billiard paths.