

Hwk #32:

This example is taken from *I. Rosenholtz, L. Smylie: "The only Critical Point in Town" Test*, *Mathematics Magazine* **58**(1985), 149–150.

Show that the function

$$g : (x, y) \mapsto y^2 + 3(y + e^x - 1)^2 + 2(y + e^x - 1)^3, \mathbb{R}^2 \rightarrow \mathbb{R}$$

has exactly one critical point, and that this point is a relative minimum.

Furthermore explain why this point is NOT an absolute minimum.

Hwk #33:

This example is taken from Marsden-Tromba: Show that the function f given by $f(x, y) = (y - 3x^2)(y - x^2)$ has a critical point in the origin, which is neither a relative minimum nor a relative maximum. What kind of ‘***’ definite is the Hessian?

Show also that all single-variable radial functions $t \mapsto f(t \cos \phi, t \sin \phi)$ have a relative minimum at $t = 0$.

Hwk #34:

This example is geometrically appealing, but alas computationally lengthy. This is why I give you the intermediate steps and hints to navigate you through. Ideally, it should be done with the help of symbolic algebra software, and you are welcome to use this tool, if available.

We want to find a shortest connection between two plane curves, namely $y = x^2 + 2$ and $y = \frac{1}{2}(x-1)^2$. A precise plot is attached. Choose points $P = (a, a^2+2)$ on the first parabola and $Q = (b, \frac{1}{2}(b-1)^2)$ on the second and minimize the square of the distance. Determine all critical points and classify them. Does one of them provide a *global* minimum? Why?

Hint 1: While it is possible to take one of the equations for a critical point and solve it for b via by means of the quadratic formula, and then plug in the result in the other equation, this is tedious. It is more straightforward to take successively linear combinations of the two equations with the strategy of first eliminating b^3 , then b^2 , then b , until one polynomial equation in a remains.

Hint 2: After an obvious factorization of this polynomial equation, an easy solution $a = 1$ can be guessed, and when this is factored off, a 4th order polynomial remains that can be factored into two quadratics with integer coefficients; indeed one factor is $a^2 + 2a + 3$.

Hwk #35:

Suppose in the following matrices, the starred entries are not known. Which of the five possibilities ‘positive definite’, ‘positive semidefinite (but not definite)’, ‘negative definite’,

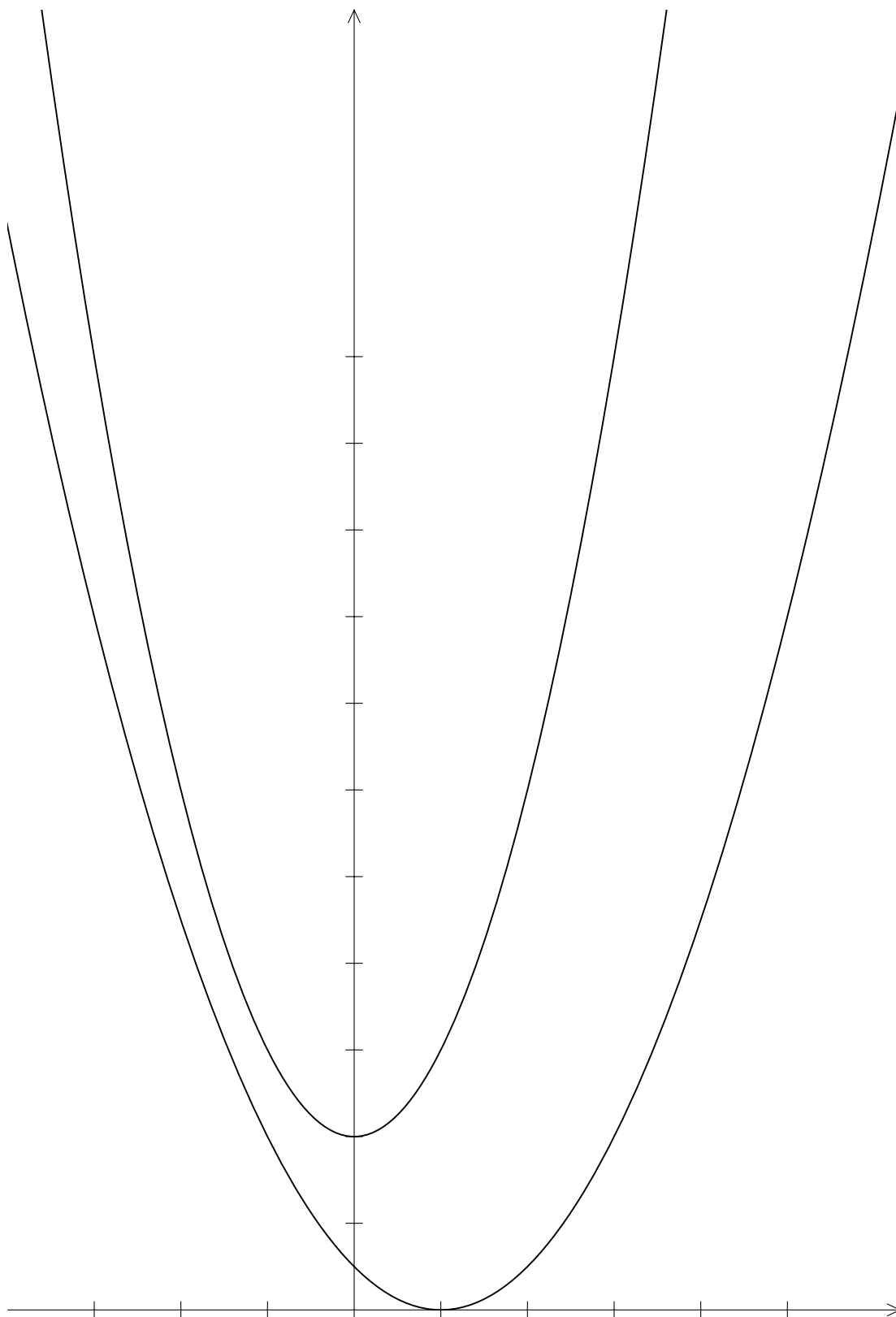


Figure 1: Figure for Hwk # 34

‘negative semidefinite (but not definite)’, ‘indefinite’ remains a possibility, based on knowledge only of the known entries?

$$(a) \begin{bmatrix} 3 & * \\ * & * \end{bmatrix} \quad (b) \begin{bmatrix} * & * \\ * & -5 \end{bmatrix} \quad (c) \begin{bmatrix} * & 6 \\ 6 & * \end{bmatrix} \quad (d) \begin{bmatrix} 3 & * \\ * & -1 \end{bmatrix} \quad (e) \begin{bmatrix} 0 & 1 & * \\ 1 & * & * \\ * & * & * \end{bmatrix}$$

Consider the following: While the Hurwitz test was worded in a way to calculate determinants starting from the top, the order in which the variables are listed (and thus determine entries of the matrix) is not essential for definiteness of a matrix; so you could use the determinants in the Hurwitz test starting at any diagonal element and then calculating 2×2 , 3×3 , etc. determinants, adding any one variable (row and column) at a time.

Hwk #36:

Find the absolute minimum and absolute maximum of $x^2 + (y - 1)^2 + z^2 - xyz$ on the ball $x^2 + y^2 + z^2 \leq 3^2$. *Hint: For the boundary consideration, use the xz plane as equator plane for the spherical coordinates, to benefit from the symmetry of the problem. Otherwise formulas get obnoxiously messy.*

Hwk #37:

Redo the boundary part of the calculations from the previous problem using Lagrange multipliers.

Hwk #38:

(a) Given two points F_1 and F_2 in the plane and a curve $f(x, y) = 0$. Let the point P be restricted to this curve and assume P is neither F_1 nor F_2 . Show: If P is such that the expression $\|F_1P\| + \|F_2P\|$ has a local maximum at P , then the path F_1PF_2 satisfies the reflection property (as defined in the problem about the ellipse). What happens in the case of a local minimum? *Note that this time the curve is arbitrary, but the point on the curve is special.*

Billiard is the French name for the game known as ‘pool’ in the US. It is also the name for the mathematician’s version of pool, regardless of country. Mathematical billiard is played on a pool table of arbitrary shape (rectangles are too boring), and it is played with a single mass point instead of balls (no spin or ‘effet’ can be given to a point, and it is immune to friction). It only bounces off the border, but not off any other mass points. Let’s assume a *strictly convex* table with smooth boundary. Strictly convex means that the straight segment connecting any two points of the boundary passes through the interior of the table. Smooth boundary means that the boundary is the level line of a smooth function with nonvanishing gradient on that level line.

(b) I claim: Given any smooth strictly convex billiard and any natural number $n \geq 2$, there is a closed n -gon billiard path. Explain why.

So far you would have been thinking we have to solve equations in order to solve minimax problems. But in this example, you argue that a minimax problem must have a solution in order to prove that some complicated equations have a solution!