Hwk #29:

(For warmup only: this is predominantly single variable calculus)
Consider $f(x) := \int_0^\infty e^{-xt}\frac{\sin t}{t}\,dt$. You will not find an immediate antiderivative by which to evaluate this integral. Nevertheless, calculate $f'(x)$. [The Math 447 expert tells you that moving the x-derivative under the t-integral is legitimate here.] You should be able to evaluate the integral that you obtain for $f'(x)$.

What do you think $\lim_{x\to\infty} f(x)$ is? [The Math 447 expert tells you that in this example it is legitimate to move $\lim_{x\to\infty}$ past the integral sign.]

Finally, knowing $f'(x)$, $\lim_{x\to\infty} f(x)$ and the fundamental theorem of calculus, find $f(x)$. Specifically, what is $\int_0^\infty \frac{\sin t}{t}\,dt$?

The late physics Nobel prize winner Richard Feynman reports in his memoirs how, as a student, he got the reputation of being an integration wizard, because he was familiar with this 'differentiation under the integral sign technique', which his peers hadn't learned.

Solution: 5 pts

$$f'(x) = \int_0^\infty \frac{\partial}{\partial x} e^{-xt}\frac{\sin t}{t}\,dt = -\int_0^\infty t e^{-xt}\frac{\sin t}{t}\,dt$$

We live on the cancelled t and evaluate the integral (e.g.) by a sequence of two integrations by parts (assuming $x > 0$). Here I derive the exponentials and integrate the trigs; the other way around would work as well.

$$\int_0^\infty e^{-xt}\sin t\,dt = [-e^{-xt}\cos t]_0^\infty - x\int_0^\infty e^{-xt}\cos t\,dt = 1 - x[e^{-xt}\sin t]_0^\infty - x^2\int_0^\infty e^{-xt}\sin t\,dt$$

By solving for the unknown integral, which was the negative of $f'(x)$, we find

$$f'(x) = -\frac{1}{1+x^2}\ .$$

Variant: Remember: if you accept complex exponentials and Euler's formula, there is another way to evaluate the same integral:

$$\int_0^\infty e^{-xt}\sin t\,dt = \frac{1}{2i}\int_0^\infty e^{-xt}(e^{it}-e^{-it})\,dt = \frac{1}{2i}\left[\frac{1}{-x+i}e^{-xt}e^{it} - \frac{1}{-x-i}e^{-xt}e^{-it}\right]_0^\infty$$
$$= \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right) = \frac{1}{x^2+1}$$

If we trust the M447 expert about moving the limit under the integral, we conclude

$$\lim_{x\to\infty} f(x) = \int_0^\infty \lim_{x\to\infty} e^{-xt}\frac{\sin t}{t}\,dt = \int_0^\infty 0\frac{\sin t}{t}\,dt = 0$$

The limit under the integral is 0 for every $t > 0$ (and we trust that the integral doesn't bother about the single exception $t = 0$; this is part of the M447 expertise on which we need to rely here for lack of deeper abstract theory).

Now we can integrate over x and argue

$$f(x_1) - f(x_0) = \int_{x_0}^{x_1} f'(x)\,dx = -\int_{x_0}^{x_1}\frac{dx}{x^2+1} = -\arctan x_1 + \arctan x_0$$

As $x_1 \to \infty$, we get $0 - f(x_0) = \arctan x_0 - \frac{\pi}{2}$. Writing x for x_0 and letting $x_0 \to 0+$ we obtain

$$\lim_{x\to 0+} f(x) = \frac{\pi}{2} - \arctan 0 = \frac{\pi}{2}$$

Note that I have been careful in writing $\lim_{x\to 0+}$ rather than plugging in $x = 0$, because our calculation of $f'(x)$ had relied on x being positive. The integral from which we calculated $f'(x)$ does NOT converge for $x = 0$! So there is really another technical issue for which we need the $M447$ expert (and it is not easy). We need to know by means of the M447 expert's theorem toolkit, that f, as defined by the integral formula, is indeed continuous (from the right) at $x = 0$. (Only) then can we conclude that $\int_0^\infty \frac{\sin t}{t}\, dt = f(0) = \frac{\pi}{2}$.

I promise you that all these theoretical gaps can be filled in, but urge you to be aware that the need to fill in the necessary theory is genuine. This type of problem is to be your motivation when you study more advanced analysis courses. On the other hand, you will see this type of calculation abundantly in physics. When it goes smoothly, physicists will not bother to justify the theoretical legitimacy. But in cases when the calculation fails (producing a wrong result because the hypotheses for legitimacy of moving limits and/or derivatives past an integral sign are not fulfilled), there is usually also an intuitive physical reason to explain why such unworried calculation is not advisable.

Hwk #30:

> Use the multi-variable chain rule to determine $f'(x)$, when $f(x) := \int_0^x \frac{\sin(xt)}{t}\, dt$.
>
> Analogous question for $g(x) := \int_{x/2}^{2x} \frac{e^{xt}}{t}\, dt$.
>
> Again, we rely on the Math 447 expert, who tells us that it is legitimate to move derivatives past the integral sign in this example.

Solution: This time, x occurs in two places in the formula for $f(x)$. The MV chain rule, written in terms of partials, tells us to consider each location separately and apply a partial (single variable) derivative (as in the single variable chain rule), and then to add the results obtained for each separate location. More formally, we consider 5 pts

$$F(u, v) := \int_0^u \frac{\sin vt}{t}\, dt$$

and we substitute $u = x$ and $v = x$ to get $f(x) = F(x, x)$. So we have

$$f'(x) = (\partial_1 F)(x, x)\frac{\partial u}{\partial x} + (\partial_2 F)(x, x)\frac{\partial v}{\partial x}$$

For $\partial_1 F$, we use the fundamental theorem (derivative of an antiderivative) to get $(\partial_1 F)(u, v) = \frac{\sin vu}{u}$. For $\partial_2 F$, we use differentiation under the integral sign (with permission from the M447 expert again given specifically for the situation of this problem, not as a blank cheque!) and get $(\partial_2 F)(u, v) = \int_0^u \frac{t \cos vt}{t}\, dt = [\frac{1}{v} \sin vt]_{t=0}^{t=u} = \frac{\sin vu}{v}$. Putting it all together (with $\partial u/\partial x = 1 = \partial v/\partial x$ because $u = x$ and $v = x$), we get

$$f'(x) = \frac{\sin(x^2)}{x} + \frac{\sin(x^2)}{x} = 2\frac{\sin(x^2)}{x}\ .$$

The same works for g, in principle: We define $G(u, v, w) := \int_w^u \frac{e^{vt}}{t}\, dt$ and let $w = x/2$, $u = 2x$, and $v = x$. So $g(x) = G(2x, x, \frac{x}{2})$. Note that derivatives with respect to the *lower* limit of integration get a minus sign from the fundamental theorem, and that we have inner derivatives $\frac{\partial w}{\partial x} = \frac{1}{2}$ and $\frac{\partial u}{\partial x} = 2$ this time.

$$g'(x) = 2\frac{e^{x\cdot 2x}}{2x} + \int_{x/2}^{2x} \frac{te^{xt}}{t}\,dt - \frac{1}{2}\frac{e^{x\cdot x/2}}{x/2} = \frac{e^{2x^2}}{x} + \left[\frac{e^{xt}}{x}\right]_{t=x/2}^{t=2x} - \frac{e^{x^2/2}}{x} = 2\frac{e^{2x^2}}{x} - 2\frac{e^{x^2/2}}{x}$$

Hwk #31:

> A quantity w depends on the coordinates x, y, z in 3-space as follows: $w = x^2 + y^2 + xyz$ (1). We study w especially on the plane given by $z = x + 2y$. Then we have there $w = x^2 + y^2 + xy(x + 2y) = x^2 + y^2 + x^2y + 2xy^2$ (2).
>
> Now we calculate $\frac{\partial w}{\partial x}$ from (1): $\frac{\partial w}{\partial x} = 2x + yz$. On the plane, this simplifies to $\frac{\partial w}{\partial x} = 2x + y(x + 2y) = 2x + xy + 2y^2$.
>
> Calculating $\frac{\partial w}{\partial x}$ on the plane directly from (2), we get $\frac{\partial w}{\partial x} = 2x+y(x+2y) = 2x+2xy+2y^2$. We clearly have a discrepancy by a term xy. What is wrong? Clear up the confusion. (This requires some text as well as formulas.)

Solution: Both calculations are correct, but the symbol $\frac{\partial w}{\partial x}$ means different things in different places, and this is the cause for the discrepancy. 5 pts

In the calculation based on (1), we take a partial derivative of the 3-variable function $f : (x, y, z) \mapsto w$. This means for $\frac{\partial}{\partial x}$ that we vary x, but keep y and z fixed. The fact that we will subsequently use the result only for specific choices of z (namely $z = x + 2y$) does not change this situation: we have calculated a directional derivative in direction $[1\,,\ 0\,,\ 0]^T$.

In the calculation based on (2), we take a partial derivative of the 2-variable composite function $g : (x, y) \mapsto w$. We have already plugged $z = x + 2y$ into the formula for f to obtain the formula for g. When we calculate $\frac{\partial}{\partial x}$, we still vary x and keep y fixed, but we do not keep z fixed any more (since $z = x + 2y$, we are changing z as we are changing x). From the perspective of the 3-variable function f, we are calculuating a directional derivative in direction $[1\,,\ 0\,,\ 1]^T$.

The chain rule clears the issue up: with $h(x, y) = x + 2y$, we have $g(x, y) = f(x, y, h(x, y))$.

$$\begin{aligned}(\partial_1 g)(x, y) = \frac{\partial g(x, y)}{\partial x} &= (\partial_1 f)(x, y, h(x, y))\frac{\partial x}{\partial x} + (\partial_2 f)(x, y, h(x, y))\frac{\partial y}{\partial x} + (\partial_3 f)(x, y, h(x, y))\frac{\partial h(x, y)}{\partial x} \\ &= (\partial_1 f)(x, y, h(x, y)) + (\partial_3 f)(x, y, h(x, y))\end{aligned}$$

The calculation based on (2) produced $(\partial_1 g)(x, y)$, whereas the calculation based on (1) produced $(\partial_1 f)(x, y, h(x, y))$. The discrepancy observed is $(\partial_3 f)(x, y, h(x, y))$.

Remember: The notation $\partial/\partial x$ only tells us which variable changes, but doesn't tell us, which variable(s) is/are held fixed.