Hwk #21:

Show that for $f(\vec{x}) := \|\vec{x}\|$ we have $\nabla f(\vec{x}) = \vec{x}/\|\vec{x}\|$ at $\vec{x} \neq \vec{0}$.

Solution: Noticing that $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots}$, we find $\partial\|\vec{x}\|/\partial x_1 = 2x_1/2\sqrt{x_1^2 + x_2^2 + \ldots}$ and similar for the other variables. So $\partial\|\vec{x}\|/\partial x_i = x_i/\|\vec{x}\|$. Putting the components in a vector, the claim is immediate. 3 pts

Also note that the function $\vec{x} \mapsto \|\vec{x}\|$ *is* differentiable outside the origin because the partials are continuous there.

Hwk #22:

Consider the vector $\vec{e}$ and any (non-empty) level set of the function $g(\vec{x}) := \|\vec{x}-\vec{e}\|+\|\vec{x}+\vec{e}\|$. We know from #13 that the level set is an ellipse. In this problem we show the reflection property of the ellipse: A ray emanating from one focus $\vec{e}$ and reflected in the ellipse will pass through the other focus $-\vec{e}$. The reflection law in physics says that the incoming ray has the same angle with the normal to a curve as the reflected ray.

Prove the reflection property of the ellipse by checking the angle between the normal to the ellipse and the ray or reflected ray respectively. You can do everything without coordinates or components, just using vector notation.

Solution: It is easy to write down a normal vector to the ellipse at point $\vec{x}$: namely, $\nabla g(\vec{x})$ is orthogonal to the ellipse, because gradients are normal to the level lines of a function. We calculate (using the previous problem) 5 pts

$$\vec{n} := \nabla g(\vec{x}) = \frac{\vec{x}-\vec{e}}{\|\vec{x}-\vec{e}\|} + \frac{\vec{x}+\vec{e}}{\|\vec{x}+\vec{e}\|}$$

Note: a use of the chain rule is implicit in this calculation, however the shift $x_i \mapsto x_i \pm e_i$ has inner dervative 1. Or, if you want to use the MV chain rule, you should work with the untransposed Df first: Take $f : \vec{x} \mapsto \vec{x} - \vec{e}$. Then $Df(\vec{x}) = I$, the unit matrix, for every $\vec{x}$. (Remember I is a matrix with 1's on the diagonal and 0's everywhere else). With $g_0(\vec{y}) := \|\vec{y}\|$, we find

$$D(g_0 \circ f)(\vec{x}) = Dg_0(f(\vec{x}))Df(\vec{x}) = (\vec{y}^T/\|\vec{y}\|)_{\vec{y}=f(\vec{x})=\vec{x}-\vec{e}} \cdot I = (\vec{x}-\vec{e})^T/\|\vec{x}-\vec{e}\|$$

Returning to our main calculation, we want to show that the angle between $\vec{x}-\vec{e}$ and $\vec{n}$ is the same as the angle between $\vec{x}+\vec{e}$ and $\vec{n}$. We check equality for the cosines of these angles, using the dot product:

$$\frac{(\vec{x}-\vec{e})\cdot\vec{n}}{\|\vec{x}-\vec{e}\|\,\|\vec{n}\|} \overset{?}{=} \frac{(\vec{x}+\vec{e})\cdot\vec{n}}{\|\vec{x}+\vec{e}\|\,\|\vec{n}\|}$$

Equivalently, we have to check

$$\frac{(\vec{x}-\vec{e})\cdot\left(\frac{\vec{x}-\vec{e}}{\|\vec{x}-\vec{e}\|} + \frac{\vec{x}+\vec{e}}{\|\vec{x}+\vec{e}\|}\right)}{\|\vec{x}-\vec{e}\|} \overset{?}{=} \frac{(\vec{x}+\vec{e})\cdot\left(\frac{\vec{x}-\vec{e}}{\|\vec{x}-\vec{e}\|} + \frac{\vec{x}+\vec{e}}{\|\vec{x}+\vec{e}\|}\right)}{\|\vec{x}+\vec{e}\|}$$

Simplifying, both sides equal

$$1 + \frac{(\vec{x}-\vec{e})\cdot(\vec{x}+\vec{e})}{\|\vec{x}-\vec{e}\|\,\|\vec{x}+\vec{e}\|}$$

and the equality of both sides proves the reflection property.

Hwk #23:

> Use the result of Hwk #15 together with the multi-variable chain rule to obtain a 'power rule' for single variable derivatives, namely $\frac{d}{dx}(f(x)^{g(x)}) =?$.
>
> Obtain the same result by single variable methods alone, writing f^g as $e^{g \ln f}$.

Solution: We consider the function $h : (u,v) \mapsto h(u,v) = u^v$, whose derivative is $Dh(u,v) = [vu^{v-1}\ ,\ u^v \ln u]$. Now we notice that $h(f(x),g(x)) = f(x)^{g(x)}$. 4 pts

By the MV chain rule,

$$\begin{aligned}\frac{d}{dx}h(f(x),g(x)) &= \frac{\partial h}{\partial u}(f(x),g(x))f'(x) + \frac{\partial h}{\partial v}(f(x),g(x))g'(x)\\ &= g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}g'(x)\ln f(x)\end{aligned}$$

Now, for comparison, the sv calculus method:

$$\frac{d}{dx}f(x)^{g(x)} = \frac{d}{dx}e^{g(x)\ln f(x)} = e^{g(x)\ln f(x)}\big(g'(x)\ln f(x) + g(x)f'(x)/f(x)\big)$$

Both results are equivalent, as they should be.

Hwk #24:

> Suppose that a piece of a level curve $g(x,y) = c$, near some point (x_0, y_0), where c is some constant, can be written as the graph of a function h: $y = h(x)$. Express the slope $h'(x_0)$ of this graph in terms of partial derivatives of g. Write your result both in mathematical notation and in physicists' notation with differential quotients (with z for the output variable of g and dy/dx expressed in terms of $\partial z/\partial x$ and $\partial z/\partial y$). *If you think a certain minus sign in your result looks weird, you're right: it does look weird, but it's still correct! Or rather I hope so, I haven't seen your solution; all I say is that the correct solution may have some weird looking detail.*

Solution: If $y = f(x)$ solves the equation $g(x,y) = c$ for y, then $g(x,f(x)) = c$, i.e., the composite function on the left hand side is actually the constant function. Taking the derivative with respect to x, we get 5 pts

$$\frac{\partial g}{\partial x}(x,f(x)) + \frac{\partial g}{\partial y}(x,f(x))f'(x) = 0$$

Note on notation: The first term in this expression looks a bit funny. In line with previous notation, we have written the partial derivative of g with respect to its first variable as $\partial g/\partial x$, because the default name for the first variable of this function was x. However, now that the arguments we plug into g are $(x,f(x))$ and therefore the variable x shows up in both variables, the notation justly raises some eyebrows, despite it being common usage. So far there is no ambiguity. The entire expression left of the equal sign, which is the (single-variable) derivative of the composite function $x \mapsto g(x,f(x))$ would be called $\frac{d}{dx}g(x,f(x))$. Nevertheless this instance hints at some possible trouble when we use default variable *names* to denote with respect to which *slot* of a multi-variable function we are taking a partial derivative. The cleaner notation would be $\partial_1 g$ instead of $\partial g/\partial x$. The index 1 tells unambiguously (and independent of the name of the variable) that we are taking a partial derivative with respect to the first variable of the function g. And once $\partial_1 g$ is constructed, we plug x in the first slot, and $f(x)$ in the second slot.

We only need to solve for $f'(x)$ yet:

$$f'(x) = -\frac{\partial g(x, f(x))/\partial x}{\partial g(x, f(x))/\partial y} = -\frac{\partial_1 g(x, f(x))}{\partial_2 g(x, f(x))}$$

Note on notation: I have just aggravated the 'default variable name identifies slot' problem in the first version, whereas the second version cleans the problem up. The way how I aggravated the problem in the first version flows out of typographic convenience: As I changed the fraction $\frac{\cdot}{\cdot}$ into $\cdot/\cdot$ to avoid towering fractions, I was forced to put the arguments $(x, f(x))$ from 'behind the formal fraction' $\frac{\partial g}{\partial x}(x, f(x))$ 'into the numerator of the formal fraction' $\frac{\partial g(x,f(x))}{\partial x}$. The first notation $\frac{\partial g}{\partial x}(x, f(x))$ at least indicates that there is a function g of which we take a certain partial derivative $\frac{\partial g}{\partial x}$ into which we then plug in some arguments. So it is manifest that the x in $\frac{\partial g}{\partial x}$ represents a slot and not a variable (b/c none is plugged in there yet). In contrast, the second notation $\frac{\partial g(x,f(x))}{\partial x}$ seems to indicate (and wrongly so) that there is an expression $g(x, f(x))$ obtained from plugging in certain x dependent quantities into the function g, and that we then take the derivative of this expression with respect to x. But this in NOT what we are doing here, and the only indication to this effect that is left is the curly ∂. For if we were really taking the derivative of $g(x, f(x))$ with respect to x, this would be a single variable derivative (as there is no other variable around any more). Likewise, if $\partial g(x, f(x))/\partial y$ were really referring to a derivative of the expression $g(x, f(x))$ with respect to y, the result would be 0, b/c there is no y in this expression any more. But in reality, this is not what is intended, and the second notation with ∂_2 and ∂_1 represents the intent so much more clearly.

Now let's look how the same formula looks in the physicist's notation: z depends on two quantities x and y. If we fix z, then y must become dependent on y (or vice versa). Our formula is written as

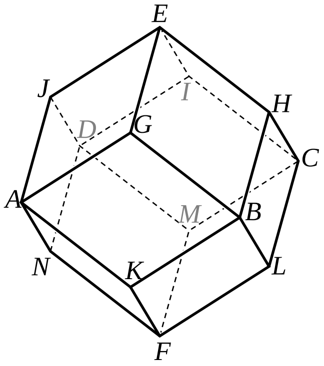
$$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}$$

If you think in terms of formally cancelling partial differentials (but you shouldn't!), then this looks paradoxical, b/c the $-$ sign doesn't seem to belong here. The physicist's version of the formula looks pretty neat and concise, but note that a lot of unspoken information is present in the context but is not represented in the formula, and it is only within this context that the formula can be interpreted meaningfully.

Variant Solution: You could also do the following: $\nabla g(x, y) = \begin{bmatrix} \partial g(x,y)/\partial x \\ \partial g(x,y)/\partial y \end{bmatrix}$ is orthogonal to the level line $g(x, y) = c$. To get a vector that is *tangent* to this level line, we need to find a vector that is orthogonal to $\nabla g(x, y)$. For instance $\begin{bmatrix} -\partial g(x,y)/\partial y \\ \partial g(x,y)/\partial x \end{bmatrix}$. (Any multiple of this vector would also qualify of course.) The slope of this tangent is $\Delta y/\Delta x = -\frac{\partial g(x,y)/\partial x}{\partial g(x,y)/\partial y}$. Since this slope if $f'(x)$, we conclude $f'(x) = -\frac{\partial g(x,y)/\partial x}{\partial g(x,y)/\partial y}$.

Hwk #25:

A rhombus is a quadrangle whose sides are all of the same length. A rhombododekahedron is a polyhedron with 12 faces, all of which are congruent rhombi. At each vertex, either four rhombi meet with their acute angles, or else, three rhombi meet with their obtuse angles. See the figure.
This does not work with an arbitrary rhombus. If you make the acute angle smaller (thus making the obtuse angle larger), the 'crown' above the zigzag $AGBHCIDJA$ becomes skinnier and taller, and the obtuse angle of the rhombus will become too large to fit below the crown as angle AGB.



Your job is to find the correct angle for a rhombus that is fit to build a rhombododekahedron. (No calculus here; just training your vector geometry and spatial vision a bit more.)

Solution: Put the origin in the center of the square $ABCD$ and the x axis parallel to AB, the y axis parallel to BC, and the z axis passing through E and F. With a convenient choice of the unit length, we have $A = (-1, -1, 0)$, $B = (1, -1, 0)$, $C = (1, 1, 0)$, $D = (-1, 1, 0)$. 6 pts

Then $E = (0, 0, e)$ and $F = (0, 0, -e)$ with e yet to be determined. (If you see that $EBFD$ should be a square as well, you know e without calculation, but I proceed to explain it assuming you don't see this.) The congruence of the rhombi requires that $\vec{EB}$ has the same length as $\vec{AB}$, namely 2. So $1^2 + 1^2 + e^2 = 2^2 + 0^2 + 0^2$. Hence $e = \sqrt{2}$.

Next $G = (0, -1, g)$ with g yet to be determined. Since the length of $\vec{AG}$ must be the same as the length of $\vec{EG}$, we conclude $1^2 + 0^2 + g^2 = 0^2 + 1^2 + (g - \sqrt{2})^2$, and therefore $g = \sqrt{2}/2$.

Now for the angle φ between the vectors $\vec{AG}$ and $\vec{AK}$, we have

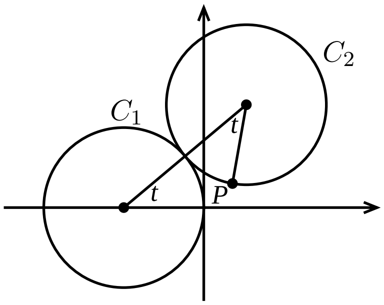
$$\cos\varphi = \vec{AG} \cdot \vec{AK} / \|\vec{AG}\| \, \|\vec{AK}\| = \begin{bmatrix} 1 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix} / (1^2 + 0^2 + (\sqrt{2}/2)^2) = \frac{1/2}{3/2} = \frac{1}{3}\,.$$

(Ok, after the tetrahedron problem and the exam problem, the mystery angle $\arccos\frac{1}{3}$ has crept up again! I hope you don't think I am obsessive compulsive about this angle;-)

Hwk #26:

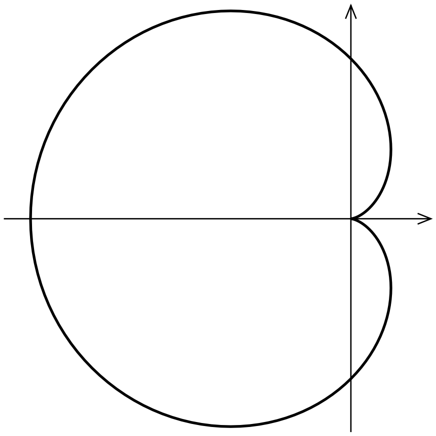
The cardioid is most easily described in terms of polar coordinates: $r = 2(1 - \cos\varphi)$. Its name comes from the Greek word for 'heart', and you'll see why when you graph this curve. So graph it carefully, but don't just steal the graph from a Valentine's card, that would be too corn(er)y!

A circle C_1 of radius 1 sits stationary with center $(-1, 0)$. Another circle C_2 of radius 1 touches it from the right, in the origin. This circle is soon to roll along the fixed circle C_1 without sliding. A point P is marked on the circle C_2. Initially it is the point where both circles touch. As C_2 rolls along C_1, the point P traces out a curve in the stationary plane. Use t for the angle (measured on C_1) of the point of contact of both circles. Give a vector valued function $t \mapsto \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ that describes the position of P as a function of t.



Show that the curve traced out by P is a cardioid.

Solution: 5 pts



The figure shows a graph of the cardioid, obtained from the formula $r = 2(1 - \cos\varphi)$.

Next we study the curve traced out by P: The center O_1 of circle C_1 has coordinates $(-1, 0)$. The vector from there to the center O_2 of circle C_2 is $[2\cos t\,,\ 2\sin t]^T$. Note that the angle O_1O_2P is also t, because C_2 rolls along C_1 without sliding. This is in addition to the angle t which the vector $\vec{O_2P}$ subtends with the horizontal. This means the vector from O_2 to P has coordinates $[-\cos 2t\,,\ -\sin 2t]^T$. We obtain the following representation of the curve traced out by P:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 + 2\cos t - \cos 2t \\ 2\sin t - \sin 2t \end{bmatrix}$$

In order to show that these two curves coincide, we rewrite the last formula, using the double-angle trig formulas:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2\cos t\,(1 - \cos t) \\ 2\sin t\,(1 - \cos t) \end{bmatrix}$$

With these formulas, it is clear that the angle φ just coincides with t, since $\tan\varphi = y/x = \tan t$; and that $r = 2(1 - \cos t)$.

Hwk #27:

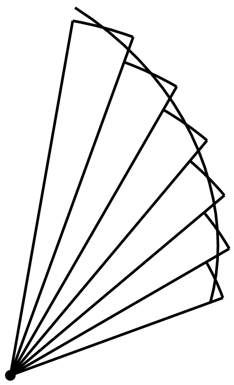
As a point P moves along a curve $r = f(\varphi)$, the line from the origin to P sweeps out a sector-like area. Approximating this area by many small sectors of a circle (a Riemann sum), find an integral formula for this area. Then use it specifically for $f(\varphi) = 2(1 - \cos\varphi)$ to find the area of the cardioid from the previous problem.

Solution: 5 pts

A sector of a circle of radius $r(\varphi)$, with angle $\Delta\varphi$, has area $\frac{1}{2}r^2\Delta\varphi$. Adding up such sectors provides a Riemann sum approximating the actual area. Therefore the area formula is $\frac{1}{2}\int_{\varphi_0}^{\varphi_1} r(\varphi)^2\,d\varphi$.

For the cardioid, this formula provides the area $A = 2\int_0^{2\pi}(1-\cos\varphi)^2\,d\varphi = 2\int_0^{2\pi}(1 - 2\cos\varphi + \cos^2\varphi)\,d\varphi$.

Evaluating the integral we get $A = 6\pi$.



Hwk #28:

Given a curve described in parametric form $t \mapsto \vec{x}(t)$, in the plane or in space, we may pretend that the parameter t represents a time and that $\vec{x}(t)$ is the position vector at 'time' t. Then the velocity is $\vec{x}'(t)$, and the speed is $\|\vec{x}'(t)\|$. The length of the curve between parameters t_0 and t_1 is $\int_{t_0}^{t_1}\|\vec{x}'(t)\|\,dt$.

Using this insight, calculate the length (perimeter) of the cardioid.

Solution: We have seen that the cardioid can be parametrized as 5 pts

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2(1-\cos t)\cos t \\ 2(1-\cos t)\sin t \end{bmatrix} = \begin{bmatrix} -1 + 2\cos t - \cos 2t \\ 2\sin t - \sin 2t \end{bmatrix}.$$

Differentiating, we get

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2\sin t + 2\sin 2t \\ 2\cos t - 2\cos 2t \end{bmatrix}$$

and hence

$$\begin{aligned} \left\| \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \right\|^2 &= (-2\sin t + 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2 \\ &= 4 + 4 - 8(\cos t\,\cos 2t + \sin t\,\sin 2t) = 8 - 8\cos t = 16\sin^2(t/2) \end{aligned}$$

Therefore the perimeter of the cardioid is

$$L = \int_0^{2\pi} 4|\sin(t/2)|\,dt = 8\int_0^{\pi}\sin s\,ds = 16$$