Hwk #15:

Find $Df(x,y)$ for $f(x,y) = x^y$, where $x > 0$.

Solution: $\partial f(x,y)/\partial x = yx^{y-1}$ (like the derivatives of x^2, x^3, etc.). 4 pts

$\partial f(x,y)/\partial y = x^y \ln x$ (like derivatives of $2^y = e^{y \ln 2}$ and the like.

Therefore $Df(x,y) = [yx^{y-1} \;\; x^y \ln x]$.

Note: *Don't go on auto-pilot from Calc2 or DiffEq: y is* NOT *a function of x here. It is the context (which functions, which dependences, are we talking about) and not some magic symbol manipulation that governs when the chain rule must be applied.*

Hwk #16:

Consider the following vector valued multi-variable functions:

$$f : (r,\varphi) \mapsto \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$

defined for $\{(r,\varphi) \mid r > 0\,,\ \varphi \in \mathbb{R}\}$; and

$$g : (r,\vartheta,\phi) \mapsto \begin{bmatrix} r\sin\vartheta\,\cos\phi \\ r\sin\vartheta\,\sin\phi \\ r\cos\vartheta \end{bmatrix}$$

defined for $r > 0$, $\vartheta \in \,]0,\pi[$, $\phi \in \mathbb{R}$.

Find $Df(t,\varphi)$ and $Dg(r,\vartheta,\phi)$ by calculating the necessary partial derivatives and observing they are continuous.

Solution: 6 pts

$$Df(r,\varphi) = \begin{bmatrix} \partial f_1(r,\varphi)/\partial r & \partial f_1(r,\varphi)/\partial\varphi \\ \partial f_2(r,\varphi)/\partial r & \partial f_2(r,\varphi)/\partial\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

$$Dg(r,\vartheta,\phi) = \begin{bmatrix} \frac{\partial g_1(r,\vartheta,\phi)}{\partial r} & \frac{\partial g_1(r,\vartheta,\phi)}{\partial\vartheta} & \frac{\partial g_1(r,\vartheta,\phi)}{\partial\phi} \\ \frac{\partial g_2(r,\vartheta,\phi)}{\partial r} & \frac{\partial g_2(r,\vartheta,\phi)}{\partial\vartheta} & \frac{\partial g_2(r,\vartheta,\phi)}{\partial\phi} \\ \frac{\partial g_3(r,\vartheta,\phi)}{\partial r} & \frac{\partial g_3(r,\vartheta,\phi)}{\partial\vartheta} & \frac{\partial g_3(r,\vartheta,\phi)}{\partial\phi} \end{bmatrix} = \begin{bmatrix} \sin\vartheta\,\cos\phi & r\cos\vartheta\,\cos\phi & -r\sin\vartheta\,\sin\phi \\ \sin\vartheta\,\sin\phi & r\cos\vartheta\,\sin\phi & r\sin\vartheta\,\cos\phi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{bmatrix}$$

Hwk #17:

Show that for any collection of real numbers $a_1, a_2, \ldots, a_n$, the following inequality is true:

$$(a_1 + a_2 + \ldots + a_n)^2 \le n(a_1^2 + a_2^2 + \ldots + a_n^2)$$

Hint: use the Cauchy Schwarz inequality on the vector $\vec{a} = [a_1, a_2, \ldots, a_n]^T$ and another vector $\vec{b}$ which you invent conveniently for the purpose.

Solution: How would we choose $\vec{b}$ wisely? Well, the CS inequality says $\vec{a}\cdot\vec{b} \le \|\vec{a}\|\,\|\vec{b}\|$. So the stuff on the smaller side of $(a_1 + a_2 + \ldots + a_n)^2 \le n(a_1^2 + a_2^2 + \ldots + a_n^2)$ should somehow match 4 pts

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \dots + a_nb_n$. This is why we choose $b_1 = b_2 = \ldots = b_n = 1$. Then, Cauchy Schwarz says:

$$|a_1 + a_2 + \ldots + a_n| = \left|\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right| \le \left\|\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right\| \left\|\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}\,\sqrt{n}$$

Squaring this immediately gives the claim.

Note: There is another, less elegant, proof: since $2ab \le a^2+b^2$, *we can argue that* $(a_1+a_2+\ldots+a_n)^2 = a_1^2 + a_2^2 + \ldots + a_n^2 + 2a_1a_2 + \ldots + 2a_1a_n + 2a_2a_3 + \ldots$. *Now we need to count the mixed terms right. For each* $2a_ia_j$ *we argue* $2a_ia_j \le a_i^2 + a_j^2$. *Each* a_i *gets thus paired wit* $(n-1)$ *other* a_j, *which accounts for* $(n-1)a_i^2$ *in the total sum.*

The same argument can be written in a more transparent way if we don't try to distinguish pure squares from mixed terms:

$$\left(\sum_{i=1}^n a_i\right)^2 = \left(\sum_{i=1}^n a_i\right)\left(\sum_{j=1}^n a_j\right) = \sum_{i=1}^n\sum_{j=1}^n a_ia_j \le \sum_{i=1}^n\sum_{j=1}^n \frac{1}{2}(a_i^2+a_j^2) = \frac{1}{2}\left(\sum_{i=1}^n na_i^2 + \sum_{j=1}^n na_j^2\right) = n\sum_{i=1}^n a_i^2$$

But of course I wanted you to see the Cauchy Schwarz inequality do something useful.

Hwk #18:

> In this problem you are to check that a certain function is differentiable *by relying on the original definition* rather than using the theorem that makes continuity of partial derivatives sufficient to guarantee differentiability.
>
> Show that $f : (x, y, z) \mapsto \sqrt{x^2 + 2y^2 + z}$ is differentiable at $(1, 1, 1)$. (No typo; it's z, not z^2, just for variety's sake.)
>
> *Strategy: as in the example in class, you use partials to find the only candidate for* $Df(1, 1, 1)$ *first; then you check the limit from the definition. — Remember: when you have a difference involving a square root in a limit problem* ($\sqrt{a} - b$ *small because* a *is close to* b^2), *you write* $\sqrt{a} - b = (a - b^2)/(\sqrt{a} + b)$. *— Another trick that may come in handy is to use the inequality in the previous problem near the end of the calculation.*

Solution: *If* a matrix T exists for which 6 pts

$$\lim_{(h,k,l)\to(0,0,0)} \frac{|f(1 + h, 1 + k, 1 + l) - f(1, 1, 1) - T[1,\ 1,\ 1]^T|}{\sqrt{h^2 + k^2 + l^2}} = 0$$

is true *then* T must be the matrix of partial derivatives

$$\left[\frac{\partial T}{\partial x}(1, 1, 1)\ ,\ \frac{\partial T}{\partial y}(1, 1, 1)\ ,\ \frac{\partial T}{\partial z}(1, 1, 1)\right]$$

Now

$$\begin{aligned}
\frac{\partial f(x, y, z)}{\partial x} &= \frac{2x}{2\sqrt{x^2 + 2y^2 + z}}\ , \qquad \frac{\partial f}{\partial x}(1, 1, 1) = \frac{1}{2} \\
\frac{\partial f(x, y, z)}{\partial y} &= \frac{4y}{2\sqrt{x^2 + 2y^2 + z}}\ , \qquad \frac{\partial f}{\partial y}(1, 1, 1) = 1 \\
\frac{\partial f(x, y, z)}{\partial z} &= \frac{1}{2\sqrt{x^2 + 2y^2 + z}}\ , \qquad \frac{\partial f}{\partial x}(1, 1, 1) = \frac{1}{4}
\end{aligned}$$

We need to estimate

$$\frac{|\sqrt{(1+h)^2+2(1+k)^2+(1+l)}-2-\frac{1}{2}h-k-\frac{1}{4}l|}{\sqrt{h^2+k^2+l^2}}$$

and show that it goes to 0 as $(h,k,l)\to(0,0,0)$.

$$\begin{aligned}
&\frac{|\sqrt{(1+h)^2+2(1+k)^2+(1+l)}-2-\frac{1}{2}h-k-\frac{1}{4}l|}{\sqrt{h^2+k^2+l^2}} = \\
&\quad = \frac{|\big((1+h)^2+2(1+k)^2+(1+l)\big)-(2+\frac{1}{2}h+k+\frac{1}{4}l)^2|}{\sqrt{h^2+k^2+l^2}\Big(\sqrt{(1+h)^2+2(1+k)^2+(1+l)}+(2+\frac{1}{2}h+k+\frac{1}{4}l)\Big)} = \\
&\quad = \frac{|\big(1+2h+h^2\ +\ 2+4k+2k^2\ +\ 1+l\big)-4-4(\frac{1}{2}h+k+\frac{1}{4}l)-(\frac{1}{2}h+k+\frac{1}{4}l)^2|}{\sqrt{h^2+k^2+l^2}\Big(\sqrt{(1+h)^2+2(1+k)^2+(1+l)}+(2+\frac{1}{2}h+k+\frac{1}{4}l)\Big)} = \\
&\quad = \frac{|h^2+2k^2-(\frac{1}{2}h+k+\frac{1}{4}l)^2|}{\sqrt{h^2+k^2+l^2}\Big(\sqrt{(1+h)^2+2(1+k)^2+(1+l)}+(2+\frac{1}{2}h+k+\frac{1}{4}l)\Big)}
\end{aligned}$$

In the first step of this calculation, we have used the identity $\sqrt{a}-b=(a-b^2)/(\sqrt{a}+b)$. The other steps only expand and cancel in the numerator.

The big parenthesi in the denominator converges to $\sqrt{1+2+1}+2=4\neq 0$ as $(h,k,l)\to(0,0,0)$. We have to show that the remaining quotient goes to 0. Indeed,

$$\frac{|h^2+2k^2-(\frac{1}{2}h+k+\frac{1}{4}l)^2|}{\sqrt{h^2+k^2+l^2}} \leq \frac{h^2+2k^2+3(\frac{1}{4}h^2+k^2+\frac{1}{16}l^2)}{\sqrt{h^2+k^2+l^2}} \leq \frac{5(h^2+k^2+l^2)}{\sqrt{h^2+k^2+l^2}} \leq 5\sqrt{h^2+k^2+l^2}$$

which clearly goes to 0.

Hwk #19:

> Assuming f differentiable, show that $\frac{d}{dt}f(\vec{x}+t\vec{v})|_{t=0}=Df(\vec{x})\vec{v}$. (The quantity on theft hand side, if it exists, is called directional derivative of f at $\vec{x}$ in direction $\vec{v}$.

Solution: We'll do it in the 3-variable case. If f is differentiable, then we know that $Df(\vec{x})$ is a row consisting of the partial derivatives. We can use, directly from the definition of total differentiability, that 5 pts

$$\frac{f(\vec{x}+t\vec{v})-f(\vec{x})-Df(\vec{x})t\vec{v}}{t\|\vec{v}\|}\to 0$$

Multiplying with $\|\vec{v}\|$, and pulling the scalar t in front of the matrix product, we get (as $t\to 0$)

$$\frac{f(\vec{x}+t\vec{v})-f(\vec{x})-t\,Df(\vec{x})\vec{v}}{t}\to 0\ ,\quad \text{i.e.,}\quad \frac{f(\vec{x}+t\vec{v})-f(\vec{x})}{t}\to Df(\vec{x})\vec{v}$$

This proves the claim.

Hwk #20:

> The multi-variable chain rule says: $D(f\circ g)(\vec{p})=Df(g(\vec{p}))Dg(\vec{p})$. Here is one specific example for which I ask you to caclulate all quantities involved in this equation and check the equality, all by explicit calculation.

$$f(x,y,z) := xyz^2 + (y-x)/(1+z^2), \quad g(\vartheta,\phi) = \begin{bmatrix} \sin\vartheta\,\cos\phi \\ \sin\vartheta\,\sin\phi \\ \cos\vartheta \end{bmatrix}.$$

(Motivation for this example: think of $f \circ g$ as a function on the unit sphere.)

Solution: 5 pts

A brief comment ahead: In this problem the $\vec{p}$ stands for $\begin{bmatrix} \vartheta \\ \phi \end{bmatrix}$. If ϑ and ϕ are indeed angle coordinates on the sphere, as suggested, then $\vec{p}$ is a non-geometric vector, artificially converted from (ϑ,ϕ). However, if ϑ and ϕ happen to be just weirdly chosen names for cartesian coordinates in a plane, then $\vec{p}$ is a geometric vector. Whichever may be the case, it does not affect the validity of the calculations.

$$(f\circ g)(\vartheta,\phi) = \sin^2\vartheta\cos^2\vartheta\sin\phi\cos\phi + \frac{\sin\vartheta(\sin\phi-\cos\phi)}{1+\cos^2\vartheta}$$

Now let's calculate all ingredients for the chain rule formula:

$$D(f\circ g)(\vartheta,\phi) = [T_{11} \quad T_{12}]$$

with

$$T_{11} = (2\sin\vartheta\cos^3\vartheta - 2\sin^3\vartheta\cos\vartheta)\sin\phi\cos\phi + \frac{\cos\vartheta(\sin\phi-\cos\phi)}{1+\cos^2\vartheta} + \frac{2\cos\vartheta\sin^2\vartheta(\sin\phi-\cos\phi)}{(1+\cos^2\vartheta)^2}$$

$$T_{12} = \sin^2\vartheta\cos^2\vartheta(\cos^2\phi-\sin^2\phi) + \frac{\sin\vartheta(\cos\phi+\sin\phi)}{1+\cos^2\vartheta}$$

$$Df(x,y,z) = [S_{11} \quad S_{12} \quad S_{13}]$$

with

$$S_{11} = yz^2 - \frac{1}{1+z^2} \qquad S_{12} = xz^2 + \frac{1}{1+z^2} \qquad S_{13} = 2xyz - \frac{2z(y-x)}{(1+z^2)^2}$$

$$Df(g(\vartheta,\phi)) = Df(\sin\vartheta\cos\phi, \sin\vartheta\sin\varphi, \cos\vartheta) = \left[\tilde{S}_{11} \quad \tilde{S}_{12} \quad \tilde{S}_{13}\right]$$

with

$$\tilde{S}_{11} = \sin\vartheta\cos^2\vartheta\sin\phi - \frac{1}{1+\cos^2\vartheta}$$

$$\tilde{S}_{12} = \sin\vartheta\cos^2\vartheta\cos\phi + \frac{1}{1+\cos^2\vartheta}$$

$$\tilde{S}_{13} = 2\sin^2\vartheta\cos\vartheta\cos\phi\sin\phi - \frac{2\cos\vartheta\sin\vartheta(\sin\phi-\cos\phi)}{(1+\cos^2\vartheta)^2}$$

$$Dg(\vartheta,\phi) = \begin{bmatrix} \cos\vartheta\,\cos\phi & -\sin\vartheta\,\sin\phi \\ \cos\vartheta\,\sin\phi & \sin\vartheta\,\cos\phi \\ -\sin\vartheta & 0 \end{bmatrix}$$

The claim of the chain rule, which we have to check here, is:

$$[T_{11} \quad T_{12}] = \left[\tilde{S}_{11} \quad \tilde{S}_{12} \quad \tilde{S}_{13}\right] \begin{bmatrix} \cos\vartheta\,\cos\phi & -\sin\vartheta\,\sin\phi \\ \cos\vartheta\,\sin\phi & \sin\vartheta\,\cos\phi \\ -\sin\vartheta & 0 \end{bmatrix}$$

In other words, the claim is

$$\begin{aligned} T_{11} &= (\tilde{S}_{11}\cos\phi + \tilde{S}_{12}\sin\phi)\,\cos\vartheta - \tilde{S}_{13}\sin\vartheta \qquad \text{and} \\ T_{12} &= (-\tilde{S}_{11}\sin\phi + \tilde{S}_{12}\cos\phi)\sin\vartheta \end{aligned}$$

and this is immediate to check.