Hwk #12:

More about the ellipse: Given the points $F_\pm = (\pm e, 0)$ in the plane (where e is some positive real number, not to be confused with the Euler number $2.718\ldots$), and a number $a > e$. Show that the set of those points $P = (x, y)$ in the plane that satisfy the condition $\|\vec{PF_+}\| + \|\vec{PF_-}\| = 2a$ is an ellipse $x^2/a^2 + y^2/b^2 = 1$. How does b relate to e and a? What is the eccentricity ε of the ellipse?

Solution: 5 pts

The vectors $\vec{PF_+}$ and $\vec{PF_-}$ are $-\begin{bmatrix} x-e \\ y \end{bmatrix}$ and $-\begin{bmatrix} x+e \\ y \end{bmatrix}$ respectively. Therefore the condition on the norms reads as

$$\sqrt{(x+e)^2 + y^2} + \sqrt{(x-e)^2 + y^2} = 2a$$

Squaring the equation and isolating the square root that remains from the mixed term yields

$$2\sqrt{(x+e)^2 + y^2}\,\sqrt{(x-e)^2 + y^2} = 4a^2 - \Big((x+e)^2 + y^2\Big) - \Big((x-e)^2 + y^2\Big)$$

Simplifying and squaring again yields

$$\Big((x+e)^2 + y^2\Big)\Big((x-e)^2 + y^2\Big) = \Big(2a^2 - (x^2 + e^2) - y^2\Big)^2$$

As we expand both sides, we benefit from a lot of cancellations:

$$\begin{aligned}
(x^2 - e^2)^2 + y^2(2x^2 + 2e^2) + y^4 &= 4a^4 + (x^2 + e^2)^2 + y^4 - 4a^2(x^2 + e^2) - 4a^2y^2 + 2(x^2 + e^2)y^2 \\
(x^2 - e^2)^2 - (x^2 + e^2)^2 &= 4a^4 - 4a^2(x^2 + e^2) - 4a^2y^2 \\
-x^2e^2 &= a^2(a^2 - x^2 - e^2 - y^2) \\
x^2(a^2 - e^2) + y^2a^2 &= a^2(a^2 - e^2) \\
\frac{x^2}{a^2} + \frac{y^2}{a^2 - e^2} &= 1
\end{aligned}$$

So we have obtained the desired equation, with $b^2 := a^2 - e^2$. — Comparing with Hwk #5, we quote:

$$a = \frac{r_0}{1 - \varepsilon^2}\ , \quad b = \frac{r_0}{\sqrt{1 - \varepsilon^2}}$$

and therefore $b^2/a^2 = 1 - \varepsilon^2$. Since $b^2 = a^2 - e^2$, we obtain $\varepsilon = e/a$. Quoting $-x_0 = \varepsilon a$ from the solution of #5, we see that $|x_0| = e$. This observation identifies the coordinate origin from #5 with the focus F_+ in the present problem.

Hwk #13:

Reconsider the function f from Problem #8: $f(x, y) := \frac{x^2y^4}{x^4+y^8}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Write it in polar coordinates: $g(r, \varphi) := f(r\cos\varphi, r\sin\varphi)$. The partial derivative $\partial g(r, \varphi)/\partial r$ at $r = 0$ is a directional derivative of f (at the origin). Show that all directional derivatives at the origin vanish, so the graph has a horizontal tangent in each direction. Nevertheless, f is not even continuous at the origin.

Plot, for some choice of fixed φ (other than an integer multiple of $\pi/2$) the graph of the single variable function $g(\cdot, \varphi) : r \mapsto g(r, \varphi)$. Include information about the precise location of the maximum of this function.

Solution: 5 pts

$$g(r,\varphi) = f(r\cos\varphi, r\sin\varphi) = \frac{r^2\cos^2\varphi\,\sin^4\varphi}{\cos^4\varphi + r^4\sin^8\varphi}$$

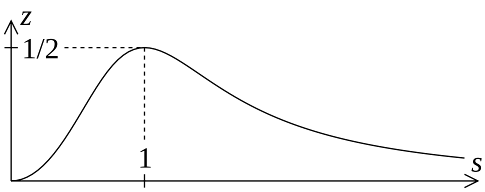
Now,

$$\frac{\partial}{\partial r} g(r,\varphi)\,|_{r=0} = \left.\frac{2r\cos^2\varphi\,\sin^4\varphi\,(\cos^4\varphi + r^4\sin^8\varphi) - r^2\cos^2\varphi\,\sin^4\varphi\,4r^3\sin^8\varphi}{(\cos^4\varphi + r^4\sin^8\varphi)^2}\right|_{r=0} = 0$$

provided $\cos\varphi \neq 0$. In the case where $\cos\varphi = 0$, we have the function $r \mapsto g(r,\varphi)$ constant 0, and the conclusion still holds.

We know from #8 that the maximum of g is $\frac{1}{2}$, and that it occurs when $r = |\cos\varphi|/\sin^2\varphi$. This can be seen by setting the r-derivative 0, or else by using the agm inequality in the denominator.

If we let $s := r\sin^2\varphi/|\cos\varphi|$, then we see that $g(r,\varphi) = s^2/(1+s^4)$, so all radial graphs arise by stretching of the s axis from one graph: $z = s^2/(1+s^4)$:



The maximum of $g(\cdot,\varphi)$ is at $r = |\cos\varphi|/\sin^2\varphi$, with value $\frac{1}{2}$. As $\varphi \to 0$, the location of this maximum moves to ∞, and near the origin, we just see the minimum. – In contrast, as $\varphi \to \pi/2$, the maximum gets closer and closer to the origin. For $\varphi = \pi/2$ exactly, the radial function is 0. This function arises as a limit from the 'decaying tail' of the graph.

Hwk #14:

> Sketch level lines for the function $f(x,y) := x^3 - 3xy^2$. Choose levels 4, 1, 0, −1, −4. The most convenient way to do this is to use polar coordinates again. Look for a trig formula involving multiple angles that fits the situation (you'd likely not have memorized this formula to recognize it at first sight, that's why I say you should look for it).
>
> *This function is hand-picked to display a rare pattern in the level lines picture*
>
> Describe the graph of f in topographer's terms: where are the hills and the valleys? The point $(0,0)$ is said to feature a *monkey saddle* of this function f.

Solution: 5 pts

$$g(r,\varphi) = f(r\cos\varphi, r\sin\varphi) = r^3(\cos^3\varphi - 3\cos\varphi\sin^2\varphi) = r^3\cos 3\varphi$$

The level curves of level h can be described in polar coordinates as $r = h^{1/3}(\cos 3\varphi)^{-1/3}$ if $h \neq 0$. The level curves for level $h = 0$ are straight lines through the origin, determined by the condition $\cos 3\varphi = 0$. As $\varphi \to \pi/6$, or any other value that makes $\cos 3\varphi$ vanish, the r coordinate on the level line goes to infinity.
This is called a monkey saddle because the monkey can sit in it facing east, with his tail hanging down west, and the legs in southeast and northeast direction.