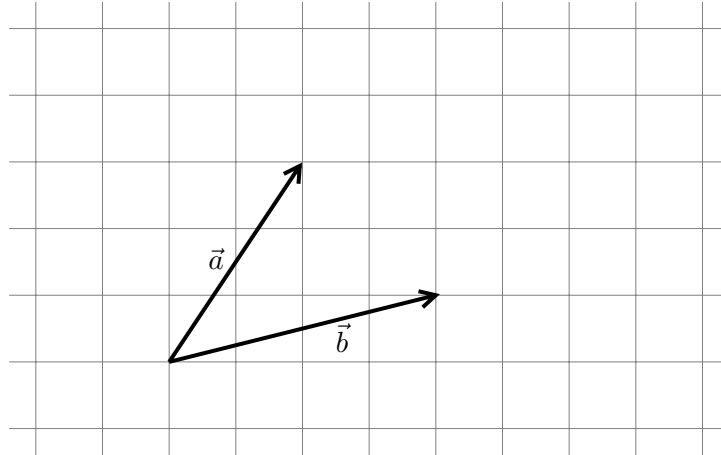


Homework  
**Math 247 – Honors Calculus 3**  
 Spring 2009 – Jochen Denzler

**Hwk #1:**

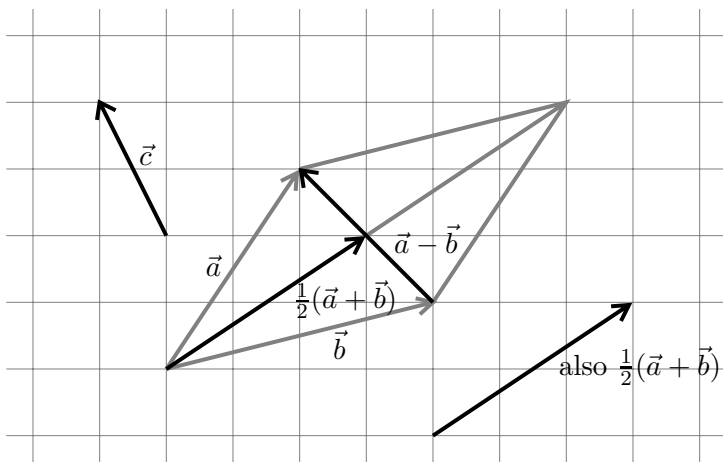
The geometric vectors  $\vec{a}$  and  $\vec{b}$  are as given in the figure below. The coordinate lines drawn are at unit distances apart. The vector  $\vec{c}$  is given in coordinates:  $\vec{c} = [-1, 2]^T$ .



- (a) Draw  $\vec{a} - \vec{b}$  and  $\frac{1}{2}(\vec{a} + \vec{b})$  into the figure.
- (b) Draw  $\vec{c} = [-1, 2]^T$  into the same figure.
- (c) Find the coordinates of  $\vec{a}$  and  $\vec{b}$ , and calculate their dot product.
- (d) Find the angle between  $\vec{a}$  and  $\vec{b}$  to a numerical precision of  $1/100^{\text{th}}$  of a degree. Since you cannot read off the coordinates from a picture to sufficient precision, I'll tell you that the coordinates of  $\vec{a}$  and  $\vec{b}$  are indeed intended to be precise integers.

**Solution:**

6 pts



$$\begin{aligned} \vec{a} &= [2, 3]^T \\ \vec{b} &= [4, 1]^T \\ \vec{a} \cdot \vec{b} &= 2 \cdot 4 + 3 \cdot 1 = 11 \end{aligned}$$

$$\cos \angle(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \frac{11}{\sqrt{2^2 + 3^2} \sqrt{4^2 + 1^2}} = \frac{11}{\sqrt{13} \cdot 17}$$

Pocket calculator:  $\varphi = \arccos \frac{11}{\sqrt{221}} = 0.73782\text{rad} = 42.27^\circ$

Note that there is no origin in this figure, and none is needed. All vectors could have been placed elsewhere, as has been done for illustration by putting a copy of  $\frac{1}{2}(\vec{a} + \vec{b})$  in a second place.

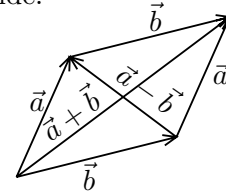
**Hwk #2:**

(a) Show that  $2\|\vec{a}\|^2 + 2\|\vec{b}\|^2 = \|\vec{a} - \vec{b}\|^2 + \|\vec{a} + \vec{b}\|^2$ . Draw a figure for illustration, and see why this formula is called the parallelogram identity.

(b) The dot product can be reconstructed from the norm: Indeed, show that  $\vec{x} \cdot \vec{y} = \frac{1}{4}(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2)$ .

**Solution:** Its easier to start evaluating these equalities from the right hand side:

$$\begin{aligned} \|\vec{a} - \vec{b}\|^2 + \|\vec{a} + \vec{b}\|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) + (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\ &= (\vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}) + (\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}) \\ &= 2\vec{a} \cdot \vec{a} + 2\vec{b} \cdot \vec{b} \\ &= 2(\|\vec{a}\|^2 + \|\vec{b}\|^2) \end{aligned}$$



4 pts

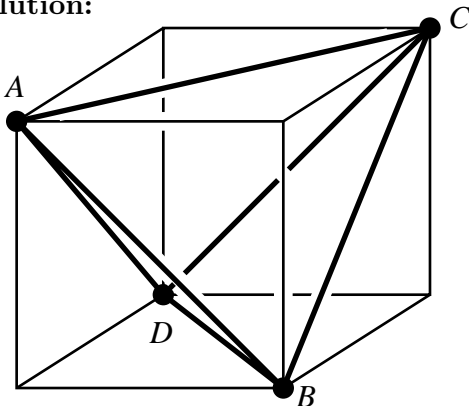
$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) - (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) \\ &= (\vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}) - (\vec{x} \cdot \vec{x} - \vec{y} \cdot \vec{x} - \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}) \\ &= 2\vec{y} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} \\ &= 4\vec{x} \cdot \vec{y} \end{aligned}$$

**Hwk #3:**

Let  $A, B, C, D$  be the vertices of a regular tetrahedron (a polyhedron whose four faces are congruent equilateral triangles). Let  $O$  be the center of this tetrahedron. Find the angle  $AOB$ . Interpretation: Chemists are interested in this angle b/c methane has a carbon atom in the center and a hydrogen atom at each of the vertices of the tetrahedron.

*Hint: Take a cube, whose center is the origin, with the axes of a coordinate system parallel to the sides of the cube; choose four of its eight vertices in a checkerboard manner: if one vertex of the cube is chosen, then the immediately adjacent ones are not and vice versa. The chosen vertices form the corners of a tetrahedron. Draw a figure. It is easy to calculate (eg) the dot product  $\vec{OA} \cdot \vec{OB}$  in terms of coordinates. The formula for this same dot product in terms of norms and angles can then be used to find the angle.*

**Solution:**



Assume the sides to be of length 2 and the origin  $O$  in the center of the cube, the  $x$  axis horizontal, the  $z$  axis vertical in the drawing plane, and the  $y$  axis perpendicular into the drawing plane. Then the coordinates are as follows:

- $A = (-1, -1, 1)$ , hence  $\vec{a} := \vec{OA} = [-1, -1, 1]^T$
- $B = (1, -1, -1)$ , hence  $\vec{b} := \vec{OB} = [1, -1, -1]^T$
- $C = (1, 1, 1)$
- $D = (-1, -1, 1)$

5 pts

Let  $\varphi$  be the angle  $AOB$ , which is the angle between  $\vec{a}$  and  $\vec{b}$ . Then  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \varphi$ . Specifically,  $-1 = \sqrt{3} \sqrt{3} \cos \varphi$ , hence  $\varphi = \arccos(-1/3) \approx 1.9106\text{rad} \approx 109.47^\circ$ .

#### Hwk #4:

(a) By copycating the proof for the Cauchy-Schwarz inequality for vectors in  $\mathbb{R}^n$ , proof that for any two continuous functions  $f, g$  on the interval  $[a, b]$ , the inequality

$$\left| \int_a^b f(x)g(x)dx \right| \leq \sqrt{\int_a^b f(x)^2 dx} \sqrt{\int_a^b g(x)^2 dx}$$

is true. (This inequality is called the Cauchy-Schwarz inequality for functions. It should be viewed as analogous to the CS inequality for vectors:  $|\vec{f} \cdot \vec{g}| \leq \|\vec{f}\| \|\vec{g}\|$ )

(b) To begin appreciating the benefit of the inequality, find out what it tells you specifically about the integral  $\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx$ , an integral you will not be able to do by means of anti-derivatives. Use the CS inequality first for  $f(x) = \sin x$  and  $g(x) = \frac{1}{x}$ , then for  $f(x) = \sqrt{\frac{\sin x}{x}}$ ,  $g(x) = \sqrt{x \sin x}$  to get a 2-sided estimate for the ‘difficult’ integral. (This technique doesn’t always give numerically as good estimates as in this particular example.)

*Note: The analogy between vectors and functions that is exploited here is studied more generally and systematically in a linear algebra course under the headings ‘Abstract real vector spaces’ and ‘Inner product spaces’*

**Solution:** (a) If  $g$  is the constant 0 on  $[a, b]$ , the inequality is obviously true. We now assume that  $g$  is not identically 0, and therefore  $\int_a^b g(x)^2 dx > 0$ . Consider the quantity  $\int_a^b (f(x) + tg(x))^2 dx$ , which is  $\geq 0$  for any real number  $t$ . We will later choose  $t$  conveniently. Expanding the square, we obtain 6 pts

$$0 \leq \int_a^b f(x)^2 dx + 2t \int_a^b f(x)g(x) dx + t^2 \int_a^b g(x)^2 dx$$

If we now let  $t := -\int_a^b f(x)g(x) dx / \int_a^b g(x)^2 dx$  (which is possible since  $\int_a^b g(x)^2 dx > 0$ ), we obtain

$$0 \leq \int_a^b f(x)^2 dx - \frac{(\int_a^b f(x)g(x) dx)^2}{\int_a^b g(x)^2 dx}$$

Clearing the denominator, moving the negative term over, and taking the square root proves the claim.

(b) For  $f(x) = \sin x$  and  $g(x) = \frac{1}{x}$ , we obtain

$$\left| \int_{\pi/2}^{\pi} \frac{\sin x}{x} dx \right| \leq \sqrt{\int_{\pi/2}^{\pi} \sin^2 x dx} \sqrt{\int_{\pi/2}^{\pi} \frac{dx}{x^2}}$$

With

$$\int_{\pi/2}^{\pi} \sin^2 x dx = \frac{1}{2} [x - \sin x \cos x]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

and

$$\int_{\pi/2}^{\pi} \frac{dx}{x^2} = \frac{1}{\pi}$$

we conclude  $\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx \leq \frac{1}{2}$ .

Next, with  $f(x) = \sqrt{\frac{\sin x}{x}}$  and  $g(x) = \sqrt{x \sin x}$ , we conclude

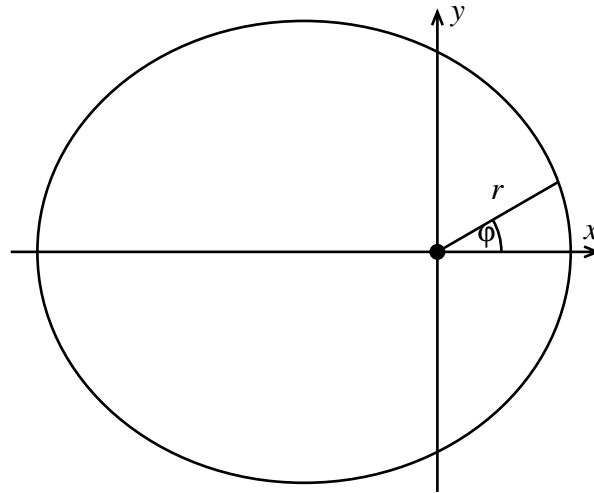
$$\left| \int_{\pi/2}^{\pi} \sin x dx \right| \leq \sqrt{\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx} \sqrt{\int_{\pi/2}^{\pi} x \sin x dx}$$

Using  $\int_{\pi/2}^{\pi} x \sin x \, dx = \pi - 1$ , and  $\int_{\pi/2}^{\pi} \sin x \, dx = 1$ , we conclude after squaring that  $\int_{\pi/2}^{\pi} \frac{\sin x}{x} \, dx \geq 1/(\pi - 1)$ .

Conclusion:  $0.4669 \leq \int_{\pi/2}^{\pi} \frac{\sin x}{x} \, dx \leq 0.5$ .

**Hwk #5:**

In astronomy, it is convenient to describe an ellipse (like, e.g., the orbit of the earth around the sun) in polar coordinates (with the sun at the origin, and the positive  $x$  axis through the perihelion (the point on the orbit of the earth that is closest to the sun)). See figure (not to scale).



In polar coordinates, the orbit is given by  $r = r_0/(1 + \varepsilon \cos \varphi)$ , where  $\varepsilon \in ]0, 1[$  is called the eccentricity (a measure how much the ellipse deviates from circular shape). Obtain the equation in cartesian coordinates  $(x, y)$ . The answer should be in the form (you fill in the ‘?’).

$$\frac{(x-?)^2}{?^2} + \frac{y^2}{?^2} = 1$$

**Solution:**

5 pts

The connection between polar and cartesian coordinates is given by  $x = r \cos \varphi$ ,  $y = r \sin \varphi$ . Therefore  $\cos \varphi = x/r$  and  $r^2 = x^2 + y^2$ .

We write  $r = r_0/(1 + \varepsilon \cos \varphi)$  as  $r + r\varepsilon \cos \varphi = r_0$ , or equivalently,  $r = r_0 - \varepsilon x$ . Squaring produces  $x^2 + y^2 = (r_0 - \varepsilon x)^2$ . Gathering like terms gives us  $x^2(1 - \varepsilon^2) + 2r_0\varepsilon x + y^2 = r_0^2$ .

After completing the square for  $x$ , we get  $(1 - \varepsilon^2)(x + \frac{r_0\varepsilon}{1-\varepsilon^2})^2 + y^2 = r_0^2 + r_0^2 \frac{\varepsilon^2}{1-\varepsilon^2}$ .

This can be written as

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where  $x_0 = -r_0\varepsilon/(1 - \varepsilon^2)$ ,  $a = r_0/(1 - \varepsilon^2)$ ,  $b = r_0/\sqrt{1 - \varepsilon^2}$ .

$(x_0, 0)$  is the center of the ellipse,  $a$  is the major semi-axis, and  $b$  is the minor semi-axis.

