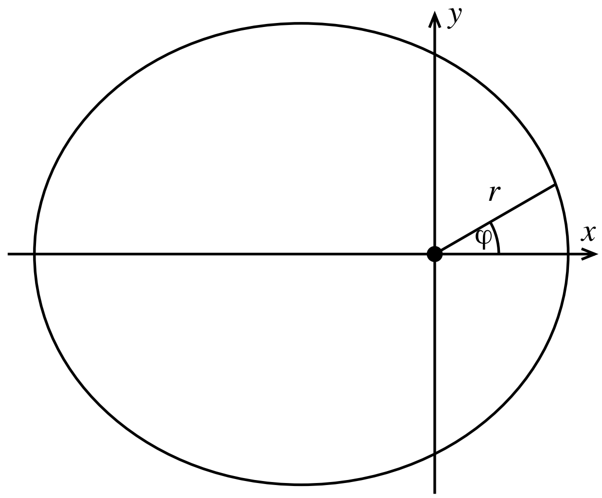
$K = (\frac{1}{2}, -\frac{1}{2}, -g)$. Again, we can get for the angle φ between the vectors $\vec{AG}$ and $\vec{AK}$ that

$$\cos\varphi = \vec{AG}\cdot\vec{AK}/\|\vec{AG}\|\,\|\vec{AK}\| = \begin{bmatrix} 1/2 \\ 1/2 \\ g \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ -g \end{bmatrix} /\Big((\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + g^2\Big) = (\tfrac{1}{2} - g^2)/(\tfrac{1}{2} + g^2) = \tfrac{1}{4}/\tfrac{3}{4} = \tfrac{1}{3}$$

Hwk #6:

In astronomy, it is convenient to describe an ellipse (like, e.g., the orbit of the earth around the sun) in polar coordinates (with the sun at the origin, and the positive x axis through the perihelion (the point on the orbit of the earth that is closest to the sun). See figure (not to scale).



In polar coordinates, the orbit is given by $r = r_0/(1 + \varepsilon\cos\varphi)$, where $\varepsilon \in \,]0, 1[$ is called the eccentricity (a measure how much the ellipse deviates from circular shape). Obtain the equation in cartesian coordinates (x, y). The answer should be in the form (you fill in the '?').

$$\frac{(x-?)^2}{?^2} + \frac{y^2}{?^2} = 1$$

Solution:

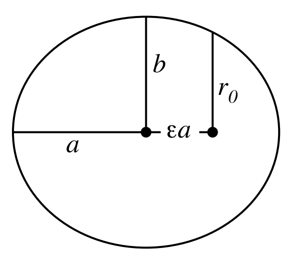
The connection between polar and cartesian coordinates is given by $x = r\cos\varphi$, $y = r\sin\varphi$. Therefore $\cos\varphi = x/r$ and $r^2 = x^2 + y^2$.

We write $r = r_0/(1 + \varepsilon\cos\varphi)$ as $r + r\varepsilon\cos\varphi_0 = r_0$, or equivalently, $r = r_0 - \varepsilon x$. Squaring produces $x^2 + y^2 = (r_0 - \varepsilon x)^2$. Gathering like terms gives us $x^2(1-\varepsilon^2) + 2r_0\varepsilon x + y^2 = r_0^2$.

After completing the square for x, we get $(1-\varepsilon^2)(x + \frac{r_0\varepsilon}{1-\varepsilon^2})^2 + y^2 = r_0^2 + r_0^2\frac{\varepsilon^2}{1-\varepsilon^2} = r_0^2/(1-\varepsilon^2)$.

This can be written as

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$



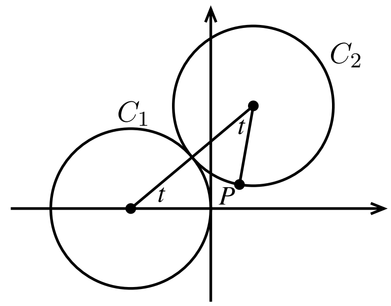
where $x_0 = -r_0\varepsilon/(1-\varepsilon^2)$, $a = r_0/(1-\varepsilon^2)$, $b = r_0/\sqrt{1-\varepsilon^2}$. $(x_0, 0)$ is the center of the ellipse, a is the major semi-axis, and b is the minor semi-axis.

Comment: The point in the allipse that was used as origin of the polar coordinate system is called a *focus* of the ellipse. The focus of the ellipse is distance εa away from the center of the ellipse. This is why ε is called the eccentricity (or excentricity).

Hwk #7:

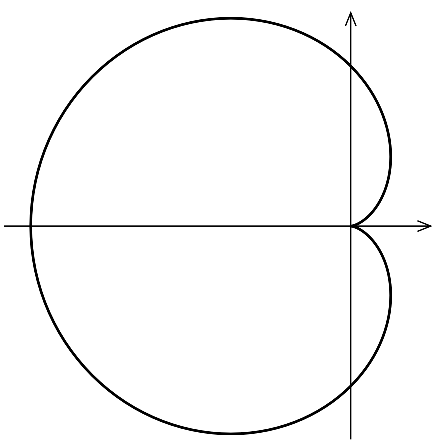
The cardioid is most easily described in terms of polar coordinates: $r = 2(1 - \cos\varphi)$. Its name comes from the Greek word for 'heart', and you'll see why when you graph this curve. So graph it carefully, but don't just steal the graph from a Valentine's card, that would be too corn(er)y!

A circle C_1 of radius 1 sits stationary with center $(-1, 0)$. Another circle C_2 of radius 1 touches it from the right, in the origin. This circle is soon to roll along the fixed circle C_1 without sliding. A point P is marked on the circle C_2. Initially it is the point where both circles touch. As C_2 rolls along C_1, the point P traces out a curve in the stationary plane. Use t for the angle (measured on C_1) of the point of contact of both circles. Give a vector valued function $t \mapsto \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ that describes the position of P as a function of t.



Show that the curve traced out by P is a cardioid.

Solution:



The figure shows a graph of the cardioid, obtained from the formula $r = 2(1 - \cos\varphi)$.

Next we study the curve traced out by P: The center O_1 of circle C_1 has coordinates $(-1, 0)$. The vector from there to the center O_2 of circle C_2 is $[2\cos t\,,\ 2\sin t]^T$. Note that the angle O_1O_2P is also t, because C_2 rolls along C_1 without sliding. This is in addition to the angle t which the vector $\vec{O_2P}$ subtends with the horizontal. This means the vector from O_2 to P has coordinates $[-\cos 2t\,,\ -\sin 2t]^T$. We obtain the following representation of the curve traced out by P:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 + 2\cos t - \cos 2t \\ 2\sin t - \sin 2t \end{bmatrix}$$

In order to show that these two curves coincide, we rewrite the last formula, using the double-angle trig formulas:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2\cos t\,(1 - \cos t) \\ 2\sin t\,(1 - \cos t) \end{bmatrix}$$

With these formulas, it is clear that the angle φ just coincides with t, since $\tan\varphi = y/x = \tan t$; and that $r = 2(1 - \cos t)$.

Hwk #8:

Given a curve described in parametric form $t \mapsto \vec{x}(t)$, in the plane or in space, we may pretend that the parameter t represents a time and that $\vec{x}(t)$ is the position vector at 'time' t. Then the velocity is $\vec{x}'(t)$, and the speed is $\|\vec{x}'(t)\|$. The length of the curve between parameters t_0 and t_1 is $\int_{t_0}^{t_1} \|\vec{x}'(t)\|\,dt$.

Using this insight, calculate the length (perimeter) of the cardioid.

Solution: We have seen that the cardioid can be parametrized as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2(1-\cos t)\cos t \\ 2(1-\cos t)\sin t \end{bmatrix} = \begin{bmatrix} -1+2\cos t-\cos 2t \\ 2\sin t-\sin 2t \end{bmatrix} .$$

Differentiating, we get

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2\sin t+2\sin 2t \\ 2\cos t-2\cos 2t \end{bmatrix}$$

and hence

$$\begin{aligned} \left\| \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \right\|^2 &= (-2\sin t+2\sin 2t)^2 + (2\cos t-2\cos 2t)^2 \\ &= 4+4-8(\cos t\,\cos 2t+\sin t\,\sin 2t) = 8-8\cos t = 16\sin^2(t/2) \end{aligned}$$

Therefore the perimeter of the cardioid is

$$L = \int_0^{2\pi} 4|\sin(t/2)|\,dt = 8\int_0^{\pi} \sin s\,ds = 16$$

Hwk #9:

I love to collect airline miles. Let's assume I fly from Atlanta (ATL) to Frankfurt, Germany (FRA). How many miles is the shortest distance? I look up the following info on a map: ATL is at 84.4° western longitude and 33.65° northern latitude. FRA is at 8.6° eastern longitude and 50.1° northern latitude. The radius of the earth is 3975 mi.

Transforming from spherical to cartesian coordinates (the equator plane is the xy plane, with the Greenwich meridian going through the x axis); and using the dot product again, I can calculate the number of miles for this trip (along the shortest route, which is the arc of a circle centered at the center of the earth and connecting from ATL to FRA).

(Note: It is of course more expedient to use symbols $\lambda_{1,2}$ and $\varphi_{1,2}$ for the coordinates first, and postpone plugging in numbers until the end.)

Solution:

Location 1 (ATL) has coordinates $\begin{bmatrix} R\cos\varphi_1\cos\lambda_1 \\ R\cos\varphi_1\sin\lambda_1 \\ R\sin\varphi_1 \end{bmatrix} =: \vec{v}_1.$

Location 2 (FRA) has coordinates $\begin{bmatrix} R\cos\varphi_2\cos\lambda_2 \\ R\cos\varphi_2\sin\lambda_2 \\ R\sin\varphi_2 \end{bmatrix} =: \vec{v}_2.$

(The points are viewed as vectors from the origin to said points.)

For the angle δ between these two vectors (which determines the distance $R\delta$ along the great circle), it holds

$$\begin{aligned} \cos\delta = \frac{\vec{v}_1\cdot\vec{v}_2}{\|\vec{v}_1\|\,\|\vec{v}_2\|} &= \cos\varphi_1\,\cos\varphi_2\,(\cos\lambda_1\,\cos\lambda_2+\sin\lambda_1\,\sin\lambda_2)+\sin\varphi_1\,\sin\varphi_2 \\ &= \cos\varphi_1\,\cos\varphi_2\,\cos(\lambda_1-\lambda_2)+\sin\varphi_1\,\sin\varphi_2 \end{aligned}$$

Specifically, with $\lambda_1 = -84.4°$, $\lambda_2 = +8.6°$, $\varphi_1 = 33.65°$, $\varphi_2 = 50.1°$, and $R = 3975$, we obtain $R\delta = 3975\arccos(0.41578) = 4539$. ($\delta$ needed in radian of course for the validity of the formula $R\delta$.)