continuous expressions). The function $A = \sqrt{s(s-a)(s-b)(s-c)}$ is continuous, so it does take on global minima and maxima.

The boundary (when one of a, b, c is 0) does not qualify for maxima, because the area must then be 0. This is geometrically expected, but can also be seen from the formula as follows: If $a = 0$ then $s(s-a)(s-b)(s-c) = \frac{s}{8}(b+c-a)(c+a-b)(a+b-c) = -\frac{s}{8}(b+c)(c-b)^2 = -\frac{s^2}{4}(c-b)^2$ which can only be ≥ 0 (to have a square root) if $b = c$, and then the expression is 0, hence the area is 0. Similarly, when $b = 0$ or $c = 0$, the area is 0. Since there are triangles with positive area available, the boundary cases where one of a, b, c is 0, cannot be absolute maxima. There are other boundary points when e.g., $a = b + c$, which makes one factor under the square root 0. But they, too, give rise to area 0.

So the absolute maximum (or maxima, if several) must be in the interior.

Now we can search for critical points by means of Lagrange multipliers, and the absolute maximum (or maxima) must be among the critical points. Rather than minimizing the positive quantity A, we minimize A^2, which gives the same solutions (when $A \geq 0$, then A is largest exactly when A^2 is largest).

So we maximize $s(s-a)(s-b)(s-c)$ under the constraint $a+b+c-2s = 0$. We get the equations

$$\begin{aligned}
\tfrac{\partial}{\partial a} s(s-a)(s-b)(s-c) &= \lambda \tfrac{\partial}{\partial a}(a+b+c-2s) \quad \text{i.e.,} \quad -s(s-b)(s-c) = \lambda \\
\tfrac{\partial}{\partial b} s(s-a)(s-b)(s-c) &= \lambda \tfrac{\partial}{\partial b}(a+b+c-2s) \quad \text{i.e.,} \quad -s(s-a)(s-c) = \lambda \\
\tfrac{\partial}{\partial c} s(s-a)(s-b)(s-c) &= \lambda \tfrac{\partial}{\partial c}(a+b+c-2s) \quad \text{i.e.,} \quad -s(s-a)(s-b) = \lambda
\end{aligned}$$

Since we may neglect cases where $s-a$ or $s-b$ or $s-c$ is 0 (as this gives area 0 and thus certainly not a maximum), we can divide the equations on the right pairwise: For instance dividing $-s(s-b)(s-c) = \lambda$ by $-s(s-a)(s-c) = \lambda$ immediately gives $\frac{s-b}{s-a} = 1$, hence $a = b$. Dividing the 2nd by the 3rd equation gives $b = c$. So the only possible maximum can happen when $a = b = c$, an equilateral triangle. Since $a+b+c = 2s$, this requires $a = b = c = \frac{2}{3}s$. Then $A^2 = s(s-a)(s-b)(s-c) = \frac{3}{2}a(\frac{1}{2}a)^3 = \frac{3}{16}a^4$.

Hwk #41:

> Let T be the set $\{(x,y) \mid 0 \leq x \leq \pi,\ 0 \leq y \leq x\}$. Draw a figure of this set. Then evaluate the integral $I := \int_T \sin x \, \sin y \, d(x,y)$ in two ways: as iterated integral in either order.
>
> *Hint: Make sure you get the limits of integration right. If any of your calculations leaves a dangling x or y in the result you sure haven't gotten the limits right. This alert applies to all MV integral problems.*

Solution: T is a triangle: . We can integrate over y first: for each fixed x, we note that y runs from 0 to x. These y integrals occur for x from 0 to π. So,

$$I = \int_0^\pi \Big(\int_0^x \sin x \, \sin y \, dy \Big) \, dx = \int_0^\pi (1 - \cos x) \sin x \, dx = 2 \ .$$

Alternatively, we can integrate over x first. Then, for each fixed y, the x-integral extends from y to π. So we have

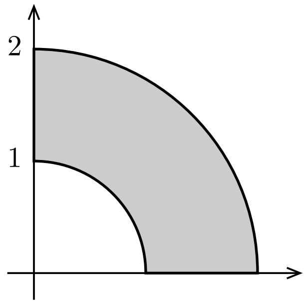
$$I = \int_0^\pi \Big(\int_y^\pi \sin x \, \sin y \, dx \Big) \, dy = \int_0^\pi (1 + \cos y) \sin y \, dy = 2 \ .$$

Hwk #42:

Let A be the set $\{(x,y) \mid 1 \le x^2+y^2 \le 4,\ x,y \ge 0\}$. Draw a figure of this set. Then evaluate the integral $I := \int_A x^2 y\, d(x,y)$ in two ways: one version using cartesian coordinates, and one using polar coordinates.

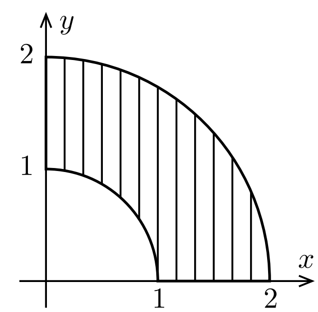
Using cartesian coordinates here is a bit dumb, admittedly. But I am asking that you do it anyways, to see the comparison with polar coordinates, and as a training to deal with the limits of integration correctly. Note that one order of integration in cartesian coordinates is easier to calculate than the other. Can you see which, and why?

Solution:



The domain is a quarter of an annulus as depicted on the left. Integration in cartesian coordinates requires splitting the outer integral into two, because the limits of integration in the inner integral are given by a piecewise function. We'll carry out both orders of integral in cartesian coordinates, for illustration purposes. But using $\int \ldots dy$ as the inner integral is easier, because the antiderivative $y^2/2$ makes the square roots disappear from the outer integral.

First we consider the integration with y as the inner integral. If $0 \le x \le 1$, the y integral extends between the two circular arcs, namely from $y = \sqrt{1-x^2}$ to $y = \sqrt{4-x^2}$. If $1 \le x \le 2$, the y integral extends from $y = 0$ to $y = \sqrt{4-x^2}$. So the outer integral needs to be split into $\int_0^1 \ldots dx + \int_1^2 \ldots dx$.



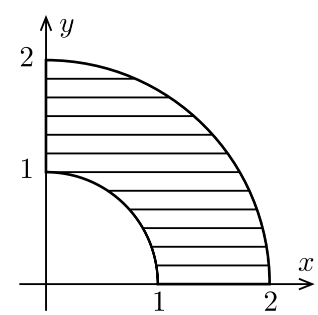
$$I = \int_0^1 \left(\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x^2 y\, dy \right) dx + \int_1^2 \left(\int_0^{\sqrt{4-x^2}} x^2 y\, dy \right) dx$$

This evaluates to

$$I = \int_0^1 x^2 \left[\frac{y^2}{2} \right]_{y=\sqrt{1-x^2}}^{y=\sqrt{4-x^2}} dx + \int_1^2 x^2 \left[\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{4-x^2}} dx = \int_0^1 \frac{3}{2} x^2\, dx + \int_1^2 \frac{1}{2} x^2 (4-x^2)\, dx = \frac{1}{2} + \frac{14}{3} - \frac{31}{10}$$

Doing the fractions we get $I = \frac{31}{15}$.

Next we consider the integration with x as the inner integral. If $0 \le y \le 1$, the x integral extends between the two circular arcs, namely from $x = \sqrt{1-y^2}$ to $x = \sqrt{4-y^2}$. If $1 \le y \le 2$, the x integral extends from $x = 0$ to $x = \sqrt{4-y^2}$. So the outer integral needs to be split into $\int_0^1 \ldots dy + \int_1^2 \ldots dy$.

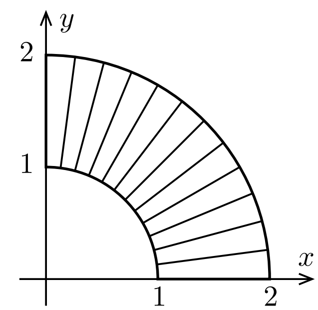


$$I = \int_0^1 \left(\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} x^2 y\, dx \right) dy + \int_1^2 \left(\int_0^{\sqrt{4-y^2}} x^2 y\, dx \right) dy$$

This evaluates to

$$
\begin{aligned}
I &= \int_0^1 y \left[\frac{x^3}{3}\right]_{x=\sqrt{1-y^2}}^{x=\sqrt{4-y^2}} dy + \int_1^2 y \left[\frac{x^3}{3}\right]_{x=0}^{x=\sqrt{4-y^2}} dy \\
&= \int_0^1 \frac{y}{3}\left((4-y^2)^{3/2} - (1-y^2)^{3/2}\right) dy + \int_1^2 \frac{y}{3}(4-y^2)^{3/2} dy \\
&= -\frac{1}{15}\left[(4-y^2)^{5/2} - (1-y^2)^{5/2}\right]_0^1 - \frac{1}{15}\left[(4-y^2)^{5/2}\right]_1^2 = \frac{4^{5/2}}{15} - \frac{1}{15} = \frac{31}{15}
\end{aligned}
$$

Finally, we evaluate the integral in polar coordinates, using $x = r\cos\varphi$, $y = r\sin\varphi$ and $d(x,y) = r\,dr\,d\varphi$. DON'T FORGET THE r IN THE DIFFERENTIAL! The limits of integration are much easier now: $1 \le r \le 2$ and $0 \le \varphi \le \frac{\pi}{2}$. Order of integration is inessential: since the integrand is a product of an r function and a φ function, both single variable integrals can be evaluated independently of each other.



$$I = \int_0^{\pi/2} \int_1^2 (r^3 \cos^2\varphi \, \sin\varphi)\, r\, dr\, d\varphi$$

We can move the trigs in front of the r-integral. Then the remaining r integral is a constant, not depending on φ, and it can therefore be moved out of the φ integral. Look carefully at this procedure: it will occur often, and at first glance it may look like a "product rule $\int fg = (\int f)(\int g)$". NOT SO! The essence is that the integrand is a product of functions each of which depends on a *different* variable, and that the limits of each integration do not depend on the other variables. It is ONLY then that you can use this trick. If you want to have a name for this rule (there doesn't seem to be an official name), call it the 'TENSOR PRODUCT RULE'.

$$I = \int_0^{\pi/2} \left(\cos^2\varphi\, \sin\varphi \int_1^2 r^4\, dr\right) d\varphi = \left(\int_1^2 r^4\, dr\right)\left(\int_0^{\pi/2} \cos^2\varphi\, \sin\varphi\, d\varphi\right) = \frac{31}{5} \cdot \frac{1}{3}[-\cos^3\varphi]_0^{\pi/2} = \frac{31}{15}$$

Hwk #43:

Physicists are familiar with the following phenomenon: If you let a massive ball and a massive cylinder roll down an incline, then the ball rolls more rapidly than the cylinder. The reason is that part of the potential energy gained when losing height is converted into kinetic energy for the forward movment, whereas another part is converted into 'internal' (rotational) kinetic energy, because the object is rolling rather than just sliding. This rotational energy is lost to the forward motion.

You may know the formula $\frac{1}{2}mv^2$ (half mass times velocity^2) for the translation energy. There is a similar formula $\frac{1}{2}I\omega^2$ for the rotation energy, where ω measures how many radians per time unit an object rotates. The quantity I is called 'moment of inertia' and it depends on the mass distribution in the body. Mass that is closer to the rotation axis counts less because it does not move as fast as mass that is farther away from the rotation axis.

The formula for I is: $I = \int_{\text{body}} s^2 \rho d\text{vol}(x, y, z)$. Here ρ is the density (which may depend on (x, y, z), but in this problem we assume it is constant). s denotes the distance from the rotation axis, which you have to express in terms of x, y, z or whatever coordinates you use.

Given this wisdom, I ask you to find I for a cylinder of radius R and height h, and also for a ball of radius R. In either case, these objects rotate about a symmetry axis. You are to express the result in the form: number times (total mass) times R^2. Remember that the total mass is volume times density ρ.

The larger the number in front of 'mass times R^2' is, the higher the proportion of energy that is used for the rotation.

Solution: If we let the ball rotate about the z axis (or rather we may say: if we choose the z axis to be the axis of rotation of the ball) and use the sherical coordiantes from Hwk #47, we note that $s = r \sin\vartheta$. So we get (with the substitution $t = -\cos\vartheta$ near the end)

$$I = \rho \int_{\text{ball}} r^2 \sin^2\vartheta \, d\text{vol} = \rho \int_0^R \int_0^\pi \int_0^{2\pi} (r^2 \sin^2\vartheta)\,(r^2 \sin\vartheta)\, d\varphi\, d\vartheta\, dr = 2\pi\rho \int_0^R \int_0^\pi r^4 \sin^3\vartheta\, d\vartheta\, dr$$
$$= 2\pi\rho \frac{R^5}{5} \int_0^\pi \sin^3\vartheta\, d\vartheta = \frac{2\pi\rho\, R^5}{5} \int_{-1}^1 (1-t^2)\, dt = \frac{2\pi\rho\, R^5}{5}\,\frac{4}{3} = \frac{2}{5}(\text{mass})R^2$$

Now for the analogous calculation for a cylinder of radius R and height h, we put the z axis along the axis of the cylinder, and we let, for instance, run z from 0 to h. (Other choices like $z \in [-\frac{h}{2}, \frac{h}{2}]$ would be just as good and lead to the same result.) We use polar coordinates in the plane, and z as a third coordinate (remember that we called these cylindrical coordinates). So we have $x = r\cos\phi$, $y = r\sin\phi$, $z = z$, and $d\text{vol} = r\, dr\, d\phi\, dz$. The distance s from the axis of rotation is r. So we calculate

$$I = \rho \int_0^R \int_0^{2\pi} \int_0^h r^2 \cdot r\, dz\, d\phi\, dr = \rho \frac{R^4}{4} \cdot 2\pi \cdot h = \frac{1}{2}(\text{mass})R^2$$

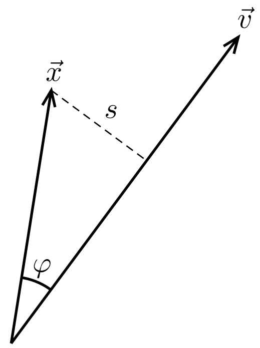
Hwk #44:

Now we rotate a cube $-a \le x, y, z \le a$ about an axis through the origin. The axis goes in the direction of a vector $\vec{v}$.

First draw a generic picture of a vector $\vec{x} = [x,\, y,\, z]^T$ and a vector $\vec{v} = [v_1,\, v_2,\, v_3]^T$ (both starting at the origin) and find a formula for the distance s of the tip of $\vec{x}$ (i.e., of the point (x, y, z)) from the axis that goes along the vector $\vec{v}$.

Then calculate the moment of inertia for this rotation (expressed as number times mass times a^2). Surprise: The final result will not depend on $\vec{v}$ — (To those who know about the tensor of inertia and the role eigenvalues play there, this surprise will be expected; but these wise folks, that's not us, for the time of Calc 3.)

Solution: Consider the following picture, in which the vectors begin in the origin:



$\vec{v}$ points along the axis of rotation, and we want the distance s of a point represented by its position vector $\vec{x}$ from this axis.

$$s^2 = \|\vec{x}\|^2 \sin^2\varphi = \|\vec{x}\|^2 - \|\vec{x}\|^2\cos^2\varphi = \|\vec{x}\|^2 - \Big((\vec{x}\cdot\vec{v})/\|\vec{v}\|\Big)^2$$

If we write $\vec{x} =: [x,\, y,\, z]^T$ and $\vec{v} =: [u,\, v,\, w]^T$ in components, we get

$$s^2 = (x^2+y^2+z^2) - \frac{(xu+yv+zw)^2}{u^2+v^2+w^2}$$

Without loss of generality, we assume density 1, so mass $= 8a^3$.

Now we have to calculate

$$\begin{aligned}
I &:= \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} s^2\,dx\,dy\,dz \\
&= \frac{2}{3}a^3(2a)^2 + \frac{2}{3}a^3(2a)^2 + \frac{2}{3}a^3(2a)^2 - \frac{1}{u^2+v^2+w^2}\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}(xu+yv+zw)^2\,dx\,dy\,dz \\
&= 8a^5 - \frac{1}{u^2+v^2+w^2}\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}(x^2u^2+y^2v^2+z^2w^2+2uvxy+2uwxz+2vwyz)\,dx\,dy\,dz \\
&= 8a^5 - \frac{1}{u^2+v^2+w^2}\left((u^2+v^2+w^2)\frac{8}{3}a^5 + (uv+uw+vw)\cdot 0\right) \\
&= \frac{16}{3}a^5 = \frac{2}{3}(\text{mass})a^2
\end{aligned}$$

Hwk #45:

Find the center of mass of the 'full' cardioid, i.e., of the area enclosed by the curve $r = a(1-\cos\varphi)$ in polar cordinates. Is it the same as the center of mass of the *curve* $r = a(1-\cos\varphi)$, which we calculated to be $(-\frac{4}{5}a, 0)$?

Solution: Let C denote the cardioid. We have to calculate

$$x_{CM} = \frac{\int_C x\,\text{darea}(x,y)}{\int_C 1\,\text{darea}(x,y)} \quad \text{and} \quad y_{CM} = \frac{\int_C y\,\text{darea}(x,y)}{\int_C 1\,\text{darea}(x,y)}$$

Of course by symmetry, we expect $y_{CM} = 0$. We express the area in polar coordinates: r ranges from 0 to $a(1-\cos\varphi)$, where φ ranges from $-\pi$ to π (or 0 to 2π if you prefer). Then $d\text{area} = r\,dr\,d\varphi$. So

$$\begin{aligned}
x_{CM} &= \frac{\int_{-\pi}^{\pi}\int_0^{a(1-\cos\varphi)} r\cos\varphi\, r\,dr\,d\varphi}{\int_{-\pi}^{\pi}\int_0^{a(1-\cos\varphi)} r\,dr\,d\varphi} = \frac{\int_{-\pi}^{\pi}\frac{1}{3}a^3(1-\cos\varphi)^3\cos\varphi\,d\varphi}{\int_{-\pi}^{\pi}\frac{1}{2}a^2(1-\cos\varphi)^2\,d\varphi} \\
&= \frac{2}{3}a\,\frac{\int_{-\pi}^{\pi}(\cos\varphi - 3\cos^2\varphi + 3\cos^3\varphi - \cos^4\varphi)\,d\varphi}{\int_{-\pi}^{\pi}(1-2\cos\varphi+\cos^2\varphi)\,d\varphi}
\end{aligned}$$

Since $\int_{-\pi}^{\pi} \cos^{\text{odd exponent}} \varphi \, d\varphi = 0$ and $\int_{-\pi}^{\pi} \cos^2 \varphi \, d\varphi = \pi$, we only need to calculate $\int_{-\pi}^{\pi} \cos^4 \varphi \, d\varphi$ yet. It is $\frac{3}{4}\pi$ (calculation will be supplied shortly). So we conclude

$$x_{CM} = \frac{2}{3} a \frac{-3\pi \cos^2 \varphi - \frac{3}{4}\pi}{2\pi + \pi} = -\frac{5}{6} a$$

Here is the calculation for $\int \cos^4 \varphi \, d\varphi$: Do an integration by parts, then use trigonometric Pythagoras:

$$\int_{-\pi}^{\pi} \cos^3 \varphi \, \cos \varphi \, d\varphi = \Big[\cos^3 \varphi \sin \varphi\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 3 \cos^2 \varphi (-\sin \varphi) \sin \varphi \, d\varphi = 3 \int_{-\pi}^{\pi} \cos^2 \varphi (1 - \cos^2 \varphi) \, d\varphi$$

In other words,

$$\int_{\pi}^{\pi} \cos^4 \varphi \, d\varphi = 3 \int_{-\pi}^{\pi} \cos^2 \varphi \, d\varphi - 3 \int_{-\pi}^{\pi} \cos^4 \varphi \, d\varphi$$

and by solving for $\int \cos^4 \varphi \, d\varphi$, we get

$$\int_{\pi}^{\pi} \cos^4 \varphi \, d\varphi = \frac{3}{4} \int_{-\pi}^{\pi} \cos^2 \varphi \, d\varphi = \frac{3}{4}\pi$$

If we calculate y_{CM} explicitly, rather than relying on intuition to get 0, we calculate

$$y_{CM} = \frac{\int_{-\pi}^{\pi} \frac{1}{3} a^3 (1 - \cos \varphi)^3 \sin \varphi \, d\varphi}{\int_{-\pi}^{\pi} \frac{1}{2} a^2 (1 - \cos \varphi)^2 \, d\varphi} = \frac{0}{\text{who cares}} = 0$$

because the integrand in the numerator is an odd function.

Hwk #46:

Find the center of mass of the semicircle $\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ where t goes from $-\pi/2$ to $\pi/2$.

Solution: The length element is $ds = \sqrt{x'(t)^2 + y'(t)^2} \, dt = a \, dt$. So

$$x_{CM} = \frac{\int_{\text{semicircle}} x \, ds}{\int_{\text{semicircle}} ds} = \frac{\int_{-\pi/2}^{\pi/2} a \cos t \, a \, dt}{\int_{-\pi/2}^{\pi/2} a \, dt} = a \frac{2}{\pi}$$

Similarly $y_{CM} = 0$ as expected from symmetry.

Hwk #47:

Integrate the vector field $[xy, \ yz, \ xz]^T$ over the curve γ parametrized by $\vec{x}(t) = [t, \ t^2, \ t^3]^T$ for $0 \le t \le 1$. (Here of course, x, y, z are the components of the vector $\vec{x}$.)

Solution: Abbreviating the vecor field as $\vec{F}$, we have

$$\int_\gamma \vec{F}(\vec{x}) \cdot d\vec{s} = \int_0^1 \vec{F}(x(t), y(t), z(t)) \cdot \vec{x}\,'(t) \, dt = \int_0^1 \begin{bmatrix} t \cdot t^2 \\ t^2 \cdot t^3 \\ t \cdot t^3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} dt = \int_0^1 (t^3 + 2t^6 + 3t^6) \, dt = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$