Hwk #19:

Consider the functions f and g given by $f(x,y) := x^2y + e^{xy} + y^3$ and $g(x,y) := \arctan\frac{y}{x}$. Calculate the following:

$$(a)\quad \frac{\partial f(x,y)}{\partial x}\,, \qquad (b)\quad \frac{\partial f(x,y)}{\partial y}\,, \qquad (c)\quad \frac{\partial g(x,y)}{\partial x}\,, \qquad (d)\quad \frac{\partial g(x,y)}{\partial y}\,,$$

Also calculate the following:

$$(a')\quad \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right), \quad (b')\quad \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right), \quad (c')\quad \frac{\partial}{\partial y}\left(\frac{\partial g(x,y)}{\partial x}\right), \quad (d')\quad \frac{\partial}{\partial x}\left(\frac{\partial g(x,y)}{\partial y}\right)$$

Compare (a') with (b') and (c') with (d').

Solution:

$$\frac{\partial f(x,y)}{\partial x} = 2xy + ye^{xy}\,, \qquad \frac{\partial f(x,y)}{\partial y} = x^2 + xe^{xy} + 3y^2$$

$$\frac{\partial g(x,y)}{\partial x} = \frac{-y/x^2}{1+(\frac{y}{x})^2} = \frac{-y}{x^2+y^2}\,, \qquad \frac{\partial g(x,y)}{\partial y} = \frac{1/x}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + ye^{xy}\right) = 2x + e^{xy} + yxe^{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 + xe^{xy} + 3y^2\right) = 2x + e^{xy} + xye^{xy}$$

So the mixed partial derivatives in either order coincide.

$$\frac{\partial}{\partial y}\left(\frac{\partial g(x,y)}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial g(x,y)}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1(x^2+y^2)-(x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Again, the mixed partial derivatives in either order coincide.

Hwk #20:

More about the ellipse: Given the points $F_\pm = (\pm e, 0)$ in the plane (where e is some positive real number, not to be confused with the Euler number $2.718\ldots$), and a number $a > e$. Show that the set of those points $P = (x,y)$ in the plane that satisfy the condition $\|\vec{PF_+}\| + \|\vec{PF_-}\| = 2a$ is an ellipse $x^2/a^2 + y^2/b^2 = 1$. How does b relate to e and a? What is the eccentricity ε of the ellipse?

Solution:
The vectors $\vec{PF_+}$ and $\vec{PF_-}$ are $-\begin{bmatrix} x-e \\ y \end{bmatrix}$ and $-\begin{bmatrix} x+e \\ y \end{bmatrix}$ respectively. Therefore the condition on the norms reads as

$$\sqrt{(x+e)^2+y^2} + \sqrt{(x-e)^2+y^2} = 2a$$

Squaring the equation and isolating the square root that remains from the mixed term yields

$$2\sqrt{(x+e)^2+y^2}\,\sqrt{(x-e)^2+y^2} = 4a^2 - \Big((x+e)^2+y^2\Big) - \Big((x-e)^2+y^2\Big)$$

Simplifying and squaring again yields

$$\Big((x+e)^2+y^2\Big)\Big((x-e)^2+y^2\Big)=\Big(2a^2-(x^2+e^2)-y^2\Big)^2$$

As we expand both sides, we benefit from a lot of cancellations:

$$\begin{aligned}(x^2-e^2)^2+y^2(2x^2+2e^2)+y^4 &= 4a^4+(x^2+e^2)^2+y^4-4a^2(x^2+e^2)-4a^2y^2+2(x^2+e^2)y^2\\ (x^2-e^2)^2-(x^2+e^2)^2 &= 4a^4-4a^2(x^2+e^2)-4a^2y^2\\ -x^2e^2 &= a^2(a^2-x^2-e^2-y^2)\\ x^2(a^2-e^2)+y^2a^2 &= a^2(a^2-e^2)\\ \frac{x^2}{a^2}+\frac{y^2}{a^2-e^2} &= 1\end{aligned}$$

So we have obtained the desired equation, with $b^2 := a^2-e^2$. — Comparing with Hwk #6, we quote:

$$a=\frac{r_0}{1-\varepsilon^2}\;,\quad b=\frac{r_0}{\sqrt{1-\varepsilon^2}}$$

and therefore $b^2/a^2 = 1-\varepsilon^2$. Since $b^2=a^2-e^2$, we obtain $\varepsilon = e/a$. Quoting $-x_0=\varepsilon a$ from the solution of #6, we see that $|x_0|=e$. This observation identifies the coordinate origin from #6 with the focus F_+ in the present problem.

Hwk #21:

Reconsider the function f from Problem #17: $f(x,y) := \frac{x^2y^4}{x^4+y^8}$ for $(x,y)\neq(0,0)$ and $f(0,0)=0$. Write it in polar coordinates: $g(r,\varphi) := f(r\cos\varphi, r\sin\varphi)$. The partial derivative $\partial g(r,\varphi)/\partial r$ at $r=0$ is a directional derivative of f (at the origin). Show that all directional derivatives at the origin vanish, so the graph has a horizontal tangent in each direction. Nevertheless, f is not even continuous at the origin.

Plot, for some choice of fixed φ (other than an integer multiple of $\pi/2$) the graph of the single variable function $g(\cdot,\varphi): r\mapsto g(r,\varphi)$. Include information about the precise location of the maximum of this function.

Solution:

$$g(r,\varphi)=f(r\cos\varphi,r\sin\varphi)=\frac{r^2\cos^2\varphi\,\sin^4\varphi}{\cos^4\varphi+r^4\sin^8\varphi}$$

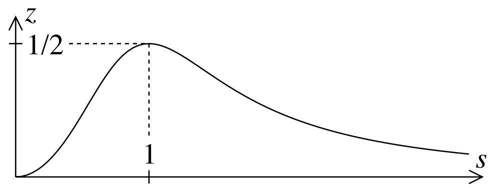
Now,

$$\frac{\partial}{\partial r}g(r,\varphi)\;|_{r=0}=\left.\frac{2r\cos^2\varphi\,\sin^4\varphi\,(\cos^4\varphi+r^4\sin^8\varphi)-r^2\cos^2\varphi\,\sin^4\varphi\,4r^3\sin^8\varphi}{(\cos^4\varphi+r^4\sin^8\varphi)^2}\right|_{r=0}=0$$

provided $\cos\varphi\neq0$. In the case where $\cos\varphi=0$, we have the function $r\mapsto g(r,\varphi)$ constant 0, and the conclusion still holds.

We know from #17 that the maximum of g is $\frac12$, and that it occurs when $r=|\cos\varphi|/\sin^2\varphi$. This can be seen by setting the r-derivative 0.

If we let $s := r\sin^2\varphi/|\cos\varphi|$, then we see that $g(r,\varphi)=s^2/(1+s^4)$, so all radial graphs arise by stretching of the s axis from one graph: $z=s^2/(1+s^4)$:



The maximum of $g(\cdot, \varphi)$ is at $r = |\cos\varphi| / \sin^2\varphi$, with value $\frac{1}{2}$. As $\varphi \to 0$, the location of this maximum moves to ∞, and near the origin, we just see the minimum. – In contrast, as $\varphi \to \pi/2$, the maximum gets closer and closer to the origin. For $\varphi = \pi/2$ exactly, the radial function is 0. This function arises as a limit from the 'decaying tail' of the graph.

Hwk #22:

Sketch level lines for the function $f(x, y) := x^3 - 3xy^2$. Choose levels 4, 1, 0, -1, -4. The most convenient way to do this is to use polar coordinates again. Look for a trig formula involving multiple angles that fits the situation (you'd likely not have memorized this formula to recognize it at first sight, that's why I say you should look for it).

This function is hand-picked to display a rare pattern in the level lines picture

Describe the graph of f in topographer's terms: where are the hills and the valleys? The point $(0, 0)$ is said to feature a *monkey saddle* of this function f.

Solution:

$$g(r, \varphi) = f(r\cos\varphi, r\sin\varphi) = r^3(\cos^3\varphi - 3\cos\varphi\sin^2\varphi) = r^3\cos 3\varphi$$

The level curves of level h can be described in polar coordinates as $r = h^{1/3}(\cos 3\varphi)^{-1/3}$ if $h \neq 0$. The level curves for level $h = 0$ are straight lines through the origin, determined by the condition $\cos 3\varphi = 0$. As $\varphi \to \pi/6$, or any other value that makes $\cos 3\varphi$ vanish, the r coordinate on the level line goes to infinity.
This is called a monkey saddle because the monkey can sit in it facing east, with his tail hanging down west, and the legs in southeast and northeast direction.