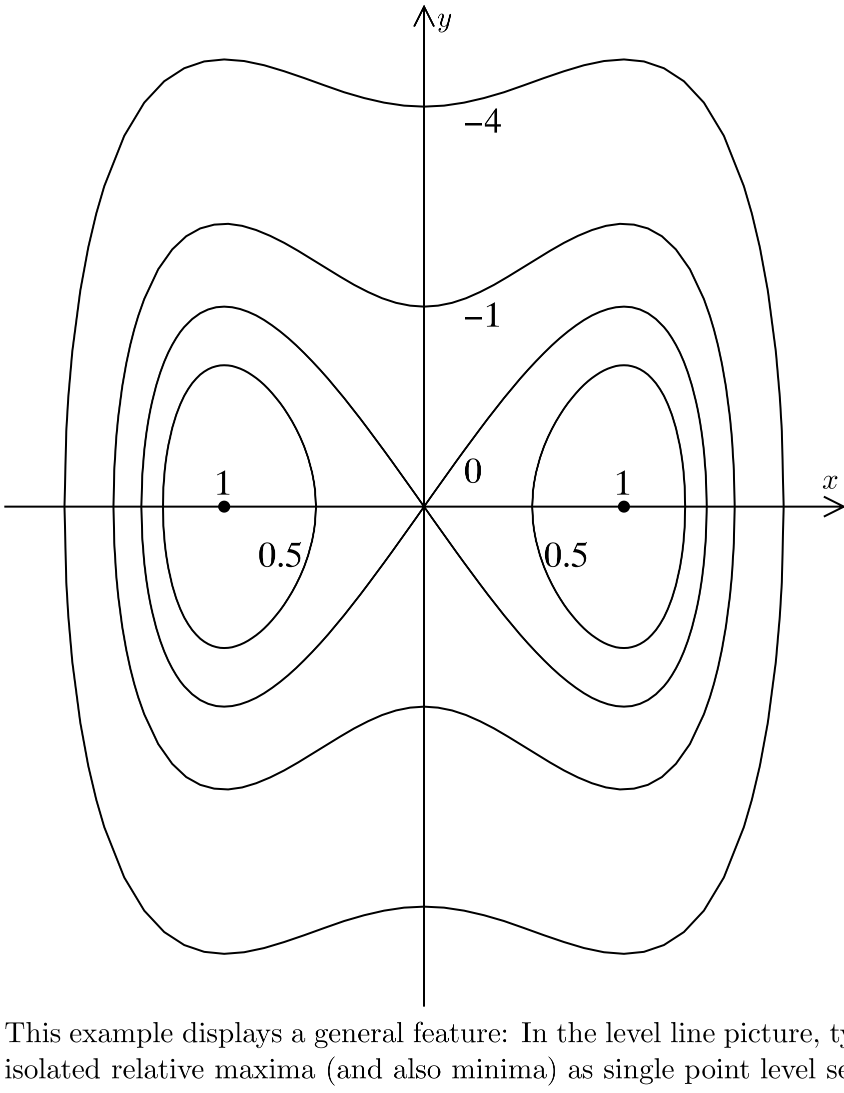
Hwk #14: *See next page. (Order swapped for page break layout reasons)*

Hwk #15:

> Try to understand the level sets of the function f given by $f(x, y) = 2x^2 - x^4 - y^2$. In particular use single-variable calculus and simple algebraic reasoning to find maxima of f. Make sure to sketch at least five level sets $\{(x, y) \mid f(x, y) = c\}$. Namely, for $c \in \{1, \frac{1}{2}, 0, -1, -4\}$. You may also find it useful to sketch graphs of a few single variable functions $g(x) := f(x, y_0)$ for various y_0. Try to avoid using technology that does 'multi-variable graphs', but feel free to enlist the help of technology for single variable graphs if this helps. The skill you are to train here is to piece single variable info together to get a multi-variable picture.

Solution:

As far as the maximum is concerned, $f(x, y) \le f(x, 0) \le f(\pm 1, 0) = 1$. Here, the single variable maximum of $g(x) := f(x, 0) = 2x^2 - x^4$ can be found by studying the derivative $g'(x) = 4(x - x^3)$ as usual; or else we can argue that $2x^2 = 2\sqrt{1 \cdot x^4} \le 1 + x^4$ by the agm inequality.



The level set for level 1 consists of only the two points $(\pm 1, 0)$. The level set for level z consists of two sv function graphs $y = \pm\sqrt{-z + 2x^2 - x^4}$. For $0 < z < 1$, the term under the root is non-negative on two intervals, namely for $x^2 \in [1 - \sqrt{1 - z}, 1 + \sqrt{1 - z}]$. These level sets are in the shape of two (slightly deformed) circles, each surrounding a maximum. For $z = 0$, the two circles merge into a figure-8 curve; for $z < 0$, the term under te square root is non-negative on a single interval centered at 0, and the level curves consist of a single closed curve.

Regarding the graph of f, the origin looks like a *saddle*. Looking only into the x direction, it looks like a minimum, but looking into the y direction, it looks like a maximum.

This example displays a general feature: In the level line picture, typical saddles show up as crossings, isolated relative maxima (and also minima) as single point level sets.

Hwk #14:

Draw level curves of the function f given by $f(x,y) := |x| + |y|$, and describe the graph $z = f(x,y)$.

Same question for $g(x,y) = \sqrt{x^2+y^2}$.

Solution: The level curves for f are squares whose vertices are on the x and y axes respectively. The graph of f is a four-sided pyramid 'standing on its tip'.

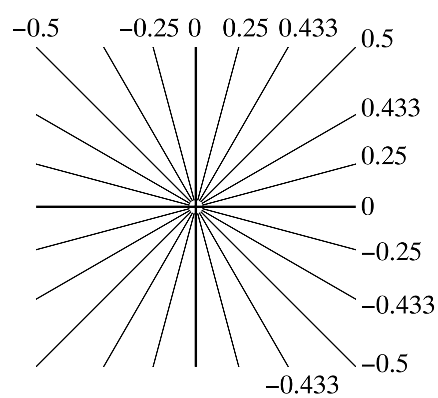
The level curves of g are circles, and the graph of g is a cone 'standing on its tip'.

Hwk #16:

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. By using polar cordinates, draw the level curves of this function. As explained in class, this function is *not* continuous at $(0,0)$. Show that nevertheless, all the single variable functions g and h given by $g(x) := f(x,y)$ for any choice of y, and $h(y) := f(x,y)$ for any choice of x, are continuous. For any fixed k, find the limit $\lim_{x\to 0} f(x,kx)$.

Solution:
We let $x = r\cos\varphi$ and $y = r\sin\varphi$ and define $\tilde{f}(r,\varphi) := f(r\cos\varphi, r\sin\varphi)$. (Note that in comparison, physicists may prefer to re-use the symbol f for the new function, with the names of the arguments being used to resolve the ambiguity. – I am using the mathematicians' convention here. Refer to the introductory notes (pg 1) for this issue)

Then $\tilde{f}(r,\varphi) := \cos\varphi\,\sin\varphi = \frac{1}{2}\sin(2\varphi)$. The level curves are straight lines 'through' (but omitting) the origin.
Obviously, the limits $\lim_{x\to 0} f(x,kx) = k/(1+k^2)$ and $\lim_{y\to 0} f(0,y) = 0$ exist.
For fixed $x \neq 0$, the sv function $h : y \mapsto \frac{xy}{x^2+y^2}$ is continuous, b/c it is a rational function with non-vanishing denominator. For $x = 0$, we are considering the constant function $h : y \mapsto 0$ (defined for every y including 0, because of the stipulation $f(0,0) = 0$). This h is trivially continuous. The analogous arguments apply to the functions $g : x \mapsto \frac{xy}{x^2+y^2}$ for fixed y and (in case $y = 0$) to $g : x \mapsto 0$.



Hwk #17:

Cranking the previous example up a notch, consider the function f given by $f(x,y) := \frac{x^2y^4}{x^4+y^8}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

Show that each of the radial limits $\lim_{x\to 0} f(x,kx)$ and $\lim_{y\to 0} f(0,y)$ equal $0 = f(0,0)$, but that, nevertheless, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (and hence f is not continuous in the origin). Draw level lines of f to get insight into this function.

Can you describe a 'curve of approach' in the (x,y) plane along which the single variable limit exists, but is different from 0? That is, can you find $x(t)$ and $y(t)$ such that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$, but $\lim_{t\to\infty} f(x(t),y(t)) \neq 0$.

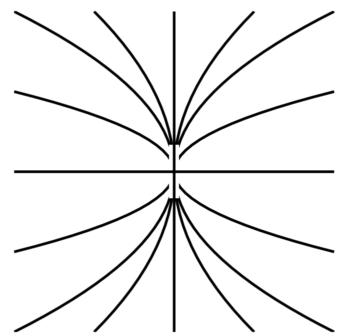
Can you also describe a curve of approach for which the limit does not exist?

Solution:

$$\lim_{x\to 0} f(x,kx) = \lim_{x\to 0} \frac{k^4x^6}{x^4+k^8x^8} = \lim_{x\to 0} \frac{k^4x^2}{1+k^8x^4} = 0$$

because the last expression under the limit is continuous. Moreover, $\lim_{y\to 0} f(0,y) = \lim_{y\to 0}(0\cdot y^4/y^8) = 0$. (Note that during the $\lim_{y\to 0}$, by definition, y is *not* 0.)

The level lines are parabolas $y^2 = kx$, for level $k^2/(1+k^4)$. For instance, if we approach the origin on such parabolas, we get different limits: $\lim_{y\to 0} f(y^2,y) = \frac{1}{2}$; or $\lim_{y\to 0} f(\frac{1}{2}y^2,y) = \frac{4}{17}$. — (The parametrization for the curve of approach suggested in the problem isn't the most convenient for these parabolas, but we could consider $(x(t),y(t)) = (1/t^2, 1/t)$ for the parabola $x = y^2$.)



The limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, because in every ball around the origin, f takes on (among others), in particular the values 0 and $\frac{1}{2}$, so no number L can be within distance $\varepsilon := 0.1$ from both 0 and $\frac{1}{2}$.

To take a curve of approach for which the limit does not exist, we may want to take some kind of spiral around the origin, that keeps encountering all level sets in turn. For instance, we might consider trying $(x(t),y(t)) := (\frac{1}{t}\cos t, \frac{1}{t}\sin t) \to (0,0)$ as $t\to\infty$, and we would get

$$f(x(t),y(t)) = \frac{\cos^2 t\,\sin^4 t}{t^2\cos^4 t + t^{-2}\sin^8 t}$$

This would work, but is technically a bit awkward to write up. We would need to select a sequence of t_k where $\cos^4 t$ is small enough to compensate for the large coefficient t^2, but not exactly 0, to prevent the numerator from being 0. We would specifically argue (using the intermediate value theorem) that there is a sequence $t_k\to\infty$ along which $t^2\cos^4 t = t^{-2}\sin^8 t$, and that for these t_k, from the equality case of the arithmetic and geometric mean inequality, the denominator equals $2\cos^2 t\sin^4 t$.

It is a bit easier to modify the curve of approach to model the different weights of x and y in the formula for f. So we can take $(x(t),y(t)) = (t^{-2}\cos^2 t\ \mathrm{sign}(\cos t), t^{-1}\sin t)$, which is still a spiral but squeezing much more in x direction than in y direction.

Then

$$f(x(t),y(t)) = \frac{\cos^4 t\,\sin^4 t}{\cos^8 t + \sin^8 t}$$

and clearly no limit exists as $t\to\infty$ (take $t_k = \frac{\pi}{4}+k\pi$ and $t'_k = k\pi$ as respective subsequences).

Hwk #18:

Does $\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$ exist, and if so, what is its value? Explain.

Solution: The limit exists and is 0 by the squeeze theorem. However, note that the domain of $y\sin\frac{1}{x}$ does not contain an entire punctured disc around the origin. So, due to the restriction on the approach direction, this limit is more akin to the single variable limits like $\lim_{x\to 0+}[\ldots]$, but obviously we cannot maintain specific notations for all the possible limitations on the approach geometry that could occur in multi-variable limits, and this is why we have put the domain hypothesis in the definition of the limit.